An Application of Hybrid of Hartley Functions for Variational Problems

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Abstract

A numerical method for solving variational problems is presented in this paper. The method is based upon hybrid of Hartley functions approximations. The properties of hybrid functions which are the combinations of block-pulse functions and Hartley functions are first presented. The operational matrix of integration is then utilized to reduce the variational problems to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamical systems. Typical examples are the Walsh functions [1], block-pulse functions [2], Laguerre polynomials [3], Legendre polynomials [4], Chebyshev series [5] and Fourier series [6]. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The approach is based on converting the underlying differential equations into integral equations through integration , approximating various involved in the equation by truncated orthogonal series and using the operational matrix of integration P , to eliminate the integral operations.

Keywords: Hybrid; Hartley functions; Operational matrix; Variational problems; Direct methods; Orthogonal functions.

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The element $h(m,n,t)$ for, $m = 1,2,...,N,\ n = -r,-r+1,...,0,1,...,r$ are the basis functions. Orthogonal on certain interval [0,1], and the matrix P can be uniquely determined on the particular orthogonal functions. The direct method of Ritz and Galerkin in solving variational problems has been of considerable concern and is well covered in many textbooks [7],[8]. Chen and Hsiao [1] introduced the Walsh series method to solve variational problems. Due to the nature of the Walsh functions, the solutions obtained were piecewise constant.

Refs.[6,9] applied Fourier series and Taylor series to obtain a solution to the second example in [1] which is an application to the heat conduction problem. It is shown that to obtain the Taylor series coefficient, a matrix commonly known as Hilbert matrix is used. Hilbert matrices are ill conditioned and hence the Taylor series is not suitable for the solution of the second example in [1].

In the present paper, we introduce a new direct computational method for solving variational problems. This method consists of reducing the variational problems to a set of algebraic equations by first expanding the candidate function as a hybrid functions [10] with unknown coefficients. These hybrid functions, which consists of block-pulse functions plus Hartley functions [11] are given. The operational matrix of integration is then utilized to evaluate the hybrid function coefficients. The variational problems are first transferred into a system of algebraic equalities. Here we will demonstrate the results by considering the illustrative examples discussed in [3] and the second example in [1].

2. Properties of Hybrid Functions

2.1. Hybrid Functions

Hybrid of Hartley functions $h(m,n,t), m = 1,2,...,N,\ n = -r,-r+1,...,-1,0,1,...,r$ defined on [0,1], have three arguments, n and m are the order for Hartley functions and block-pulse functions respectively and t is the normalized time and is defined as

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$$h(m,n,t)=\begin{cases}\phi_n(Nt+1-m), & \frac{m-1}{N}\le t\le\frac{m}{N}\\ 0, & \text{otherwise,}\end{cases} \tag{1}$$

Here, $\phi_n(t)=cas(nt)$ are the well-known Hartley functions of order n which are orthogonal in the interval [0,1] and satisfy the following formula [11]:

$$cas(nt)=Cos(2n\pi t)+Sin(2n\pi t). \tag{2}$$

Since $h(m,n,t)$ is the combination of Hartley functions and block-pulse functions, which are both complete, thus the set of hybrid functions, form a complete set.

2.2. Function Approximation

A function $f(t)$ defined over [0,1) may be expanded as

$$f(t)=\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}c_{mn}h(m,n,t), \tag{3}$$

where c_{mn} are given by

$$c_{mn}=\frac{(f(t),h(m,n,t))}{\|h(m,n,t)\|^2}, \tag{4}$$

and (.,.) denotes the inner product. If the infinite series in *Eq. (3)* is truncated, then *Eq. (3)* can be written as

$$f(t)\simeq\sum_{n=-r}^{r}\sum_{m=1}^{N}c_{mn}h(m,n,t)=C^TH(t), \tag{5}$$

where C and $H(t)$ are $(2r+1)N\times 1$ matrices given by

$$C=[c_{1,-r},\dots,c_{N,-r}\mid c_{1,-r+1},\dots,c_{N,-r+1}\mid\dots\mid c_{1,0},\dots,c_{N,0}\mid\dots\mid c_{1,r},\dots,c_{N,r}]^T, \tag{6}$$

and

$$\begin{aligned}H(t)=[&h(1,-r,t),\dots,h(N,-r,t)\mid h(1,-r+1,t),\dots,h(N,-r+1,t)\mid\\ &\dots\mid h(1,0,t),\dots,h(N,0,t)\mid\dots\mid h(1,r,t),\dots,h(N,r,t)]^T.\end{aligned} \tag{7}$$

Also, the integration of the cross product of two hybrid vector is

$$\int_0^1 H(t)H^T(t)dt=D, \tag{8}$$

where D is given by

$$D=\frac{1}{N}diag.(I,I,\dots,I),$$

and, I is an $N\times N$ identity matrix.

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2.3. Operational Matrix of Integration

The integration of the vector $H(t)$ defined in $Eq.(7)$ is given by

$$\int_0^t H(t')dt' \simeq PH(t), \tag{9}$$

where P is the $N(2r+1)\times N(2r+1)$ operational matrix for integration and is given by

$$P=\frac{1}{2N\pi}\begin{pmatrix} 0 & \dots & 0 & 0 & \frac{-I}{r} & 0 & 0 & \dots & 0 & \frac{I}{r} \\ 0 & \dots & 0 & 0 & \frac{-I}{r-1} & 0 & 0 & \dots & \frac{I}{r-1} & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \frac{-I}{2} & 0 & \frac{I}{2} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & -I & I & 0 & \dots & 0 & 0 \\ \frac{I}{r} & \dots & \frac{I}{2} & I & 2\pi P_b & \frac{-I}{r} & \frac{-I}{r-1} & \dots & \frac{-I}{2} & -I \\ 0 & \dots & 0 & -I & I & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & \frac{-I}{2} & 0 & \frac{I}{2} & 0 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{-I}{r} & \dots & 0 & 0 & \frac{I}{r} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{10}$$

In $Eq.(10)$ I is an $N\times N$ identity matrix and

$$P_b=\begin{pmatrix} \frac{1}{2} & 1 & \dots & 1 & 1 \\ 0 & \frac{1}{2} & \dots & 1 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{2} & 1 \\ 0 & 0 & \dots & 0 & \frac{1}{2} \end{pmatrix}. \tag{11}$$

3. Hybrid Functions Direct Method

Consider the problem of finding the extremum of the functional

$$J=\int_0^1 F(t,x(t),\dot{x}(t))dt. \tag{12}$$

The necessary condition for $x(t)$ to extremize $J(x)$ is that it should satisfy the Euler - Lagrange equation

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)=0,$$

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus numerical and direct method such as well-

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known Ritz and Galerkin methods have been developed to solve variational problems. Here we consider a Ritz direct method for solving *Eq.(*12*)* using hybrid functions. Suppose, the rate variable $\dot{x}(t)$ can be expressed approximately as

$$\dot{x}(t) = \sum_{n=-r}^{r} \sum_{m=1}^{N} c_{mn} h(m,n,t) = C^T H(t). \tag{13}$$

Using *Eq.(*9*)*, $x(t)$ can be represented as

$$\begin{aligned} x(t) &= \int_0^t \dot{x}(t')dt' + x(0) = C^T \int_0^t H(t')dt' + x(0) \\ &= C^T PH(t) + X_0^T H(t) = (C^T P + X_0^T)H(t), \end{aligned} \tag{14}$$

where

$$X_0 = [0,...,0 \mid 0,...,0 \mid ... \mid x(0),...,x(0) \mid 0,...,0 \mid ... \mid 0,...,0]^T.$$

We can also express t in terms of $H(t)$ as

$$\begin{aligned} t = [&\frac{1}{2rN\pi},...,\frac{1}{2rN\pi} \mid \frac{1}{2(r-1)N\pi},...,\frac{1}{2(r-1)N\pi} \mid ... \mid \frac{1}{2N},\frac{3}{2N},...,\frac{2N-1}{2N} \mid \\ &\frac{-1}{2N\pi},...,\frac{-1}{2N\pi} \mid ... \mid \frac{-1}{2rN\pi},...,\frac{-1}{2rN\pi}]H(t) = d^T H(t). \end{aligned} \tag{15}$$

Substituting *Eqs.(*13*)-(*15*)* in *Eq.(*12*)* the functional $J(x)$ becomes a function of $c_{mn}, m = 1,2,...,N, n = -r,-r+1,...,-1,0,1,...,r.$ Hence, to find the extremum of $J(x)$ we find

$$\frac{\partial J}{\partial c_{mn}} = 0. \tag{16}$$

The above procedure is now used to solve the following variational problems.

4. Illustrative Examples

4.1 Example 1

Consider the problem of finding the minimum of the functional[12]

$$J(x) = \int_0^1 [\dot{x}^2 + t\dot{x} + x^2]dt, \tag{17}$$

with boundary conditions

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$$x(0)=0, \quad x(1)=\frac{1}{4}. \tag{18}$$

Using *Eqs.(*13*)-(*15*)* in *Eq.(*17*)*, we get

$$J(x)=\int_0^1\left[C^T H(t)H^T(t)C+C^T H(t)H^T(t)d+C^T PH(t)H^T(t)P^T C\right]dt.$$

Using *Eq.(*8*)*, we obtain

$$J(x)=C^T DC+C^T Dd+C^T PDP^T C. \tag{19}$$

Also, using *Eq.(*14*)* and the boundary conditions in *Eq.(*18), we obtain

$$x(1)=C^T\int_0^1 H(t)dt=\frac{1}{4}.$$

Let

$$v=\int_0^1 H(t)dt,$$

Hence we have

$$C^T v=\frac{1}{4}. \tag{20}$$

We now minimize *Eq.(*19*)* subject to *Eq.(*20*)*, using the Lagrange multiplier technique. Suppose

$$\tilde{J}(x)=J(x)+\lambda(C^T v-\frac{1}{4}),$$

where λ is the Lagrange multiplier. Using *Eq.(*16*)*, we get

$$\frac{\partial\tilde{J}}{\partial C}=0, \quad \frac{\partial\tilde{J}}{\partial\lambda}=0$$

or

$$2DC+Dd+2PDP^T C+\lambda v=0, \quad C^T v=\frac{1}{4}. \tag{21}$$

By choosing $N,r,Eq.(21)$ is solved from which the coefficient vector C and the Lagrange multiplier λ can be found. In table 1, a comparison is made between the exact solution together with the approximate values using the present approach, for $N=2,r=2$. Since $\int_0^t H(t')dt'$ is a continuous vector we get a continuous solution for $x(t)$.

$$x(t)=C^T\int_0^t H(t')dt',$$

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Table 1. Estimated and exact values of *x(t)*

t	Exact	Hybrid functions *N=2, r = 2*
0	0	0
0.1	0.04195	0.04194
0.2	0.07932	0.07932
0.3	0.11247	0.11242
0.4	0.14175	0.14177
0.5	0.16744	0.16745
0.6	0.18981	0.18982
0.7	0.20907	0.20902
0.8	0.22541	0.22547
0.9	0.23901	0.23902
1	0.25	0.25

4.2 Example 2. Application to the heat conduction problem

Consider the extremization of

$$J(x) = \int_0^1 [\tfrac{1}{2}\dot{x}^2(t) - xg(t)]\,dt, \tag{22}$$

where $g(t)$ is a known function satisfying

$$\int_0^1 g(t)dt = -1\,, \tag{23}$$

with the boundary conditions

$$\dot{x}(0) = 0, \quad \dot{x}(1) = 0. \tag{24}$$

Schechter[13] gave a physical interpretation for this problem by noting an application in heat conduction and [1] considered the case where $g(t)$ is given by

$$g(t) = \begin{cases} -1, & 0 \le t \le \frac{1}{4}, \frac{1}{2} \le t \le 1 \\ 3, & \text{otherwise} \end{cases} \tag{25}$$

and gave and approximate solution using Walsh function. The exact solution is

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$$x(t)=\begin{cases}\frac{1}{2}t^2, & 0\le t\le\frac{1}{4}\\ -\frac{3}{2}t^2+t-\frac{1}{8}, & \frac{1}{4}<t\le\frac{1}{2}\\ \frac{1}{2}t^2-t+\frac{3}{8}, & \text{otherwise}\end{cases}$$

Here, we solve the same problem using hybrid functions. First we assume

$$\dot{x}(t)=C^T H(t).$$

In view of *Eq.(*25*)*, we write *Eq.(*22*)* as

$$J=\frac{1}{2}\int_0^1 \dot{x}^2(t)dt+4\int_0^{\frac{1}{4}}x(t)dt-4\int_0^{\frac{1}{2}}x(t)dt+\int_0^1 x(t)dt,$$

or

$$J=\frac{1}{2}\int_0^1 C^T H(t)H^T(t)Cdt+4C^T P\int_0^{\frac{1}{4}}H(t)dt-4C^T P\int_0^{\frac{1}{2}}H(t)dt+C^T P\int_0^1 H(t)dt.$$

Let

$$v(t)=\int_0^t h(t')dt'$$

then, using *Eq.(*8*)* we get

$$J=\frac{1}{2}C^T DC+C^T P[4v(\frac{1}{4})-4v(\frac{1}{2})+v(1)]. \tag{26}$$

The boundary conditions in *Eq.(*24*)* can be expressed in terms of hybrid functions as

$$C^T H(0)=0,\quad C^T H(1)=0. \tag{27}$$

We now minimize *Eq.(*26*)* subject to *Eq.(*27*)* using the Lagrange multiplier technique . Suppose

$$J^*=J+\lambda_1 C^T H(0)+\lambda_2 C^T H(1),$$

where λ_1 and λ_2 are the two multipliers.Using *Eq.(*16*)* we get

$$\frac{\partial J^*}{\partial C}=DC+P[4v(\frac{1}{4})-4v(\frac{1}{2})+v(1)]+\lambda_1 H(0)+\lambda_2 H(1)=0, \tag{28}$$

for $N=2, r=2,$ we have

$$v(\frac{1}{4})=\left[0,0\,|\,\frac{-1}{2\pi},0\,|\,\frac{1}{4},0\,|\,\frac{1}{2\pi},0\,|\,0,0\right]^T,$$

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$$v(\frac{1}{2}) = \left[0,0 \mid 0,0 \mid \frac{1}{2},0 \mid 0,0 \mid 0,0 \right]^T,$$

$$v(1) = \left[0,0 \mid 0,0 \mid \frac{1}{2},\frac{1}{2} \mid 0,0 \mid 0,0 \right]^T.$$

The computational results for $N = 2, r = 2$ together with the exact solution $x(t)$ are given in table (2).

Table 2. Estimated and exact values of *x(t)*.

t	Exact	Hybrid functions *N=2, r = 2*
0	0	0
0.1	0.005	0.00512
0.2	0.02	0.02014
0.3	0.04	0.04012
0.4	0.035	0.03499
0.5	0	-0.00007
0.6	-0.045	-0.04512
0.7	-0.08	-0.08001
0.8	-0.105	-0.10477
0.9	-0.12	-0.12012
1	-0.125	-0.12015

5. Conclusion

In the present study, the hybrid functions, which are the combinations of block-pulse functions, and Hartley functions are used to solve variational problems. The problem has been reduced to a problem of solving a system of algebraic equations. The integration of the cross product of two hybrid function vectors is a diagonal matrix, hence making hybrid functions computationally very attractive.

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