

Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R -module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper $(R, \mathfrak{m})$ is a commutative Noetherian local ring, $\mathfrak{a}$ an ideal of R and M is a finitely generated R -module. For an integer $i \in \mathbb{N}_0$, $H_{\mathfrak{a}}^i(N)$ denotes the i -th local cohomology module of M with respect to $\mathfrak{a}$ as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules $\{H_{\mathfrak{m}}^i\left(\frac{M}{\mathfrak{a}^n M}\right)\}_{n \in \mathbb{N}}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms; this family forms an inverse system. Schenzel introduced the i -th formal local cohomology of M with respect to $\mathfrak{a}$ in the form of $f_{\mathfrak{a}}^i(M) := \varprojlim_{n \in \mathbb{N}} H_{\mathfrak{m}}^i\left(\frac{M}{\mathfrak{a}^n M}\right)$, which is the i -th cohomology module of the $\mathfrak{a}$ -adic completion of the Čech complex $\check{c}_{\underline{x}} \otimes_R M$, where $\underline{x}$ denotes a system of elements of R such that $\text{Rad}(\underline{x}, R) = \mathfrak{m}$ (see [3, Definition 3.1]). He defines the formal grade as $f.\text{grade}(\mathfrak{a}, M) = \inf \{i \in \mathbb{N}_0 \mid f_{\mathfrak{a}}^i(M) \neq 0\}$. For any ideal $\mathfrak{a}$ of R and finitely generated R -module M the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of finitely generated R -modules, then there is the following long exact sequence:

$$\cdots \rightarrow f_{\mathfrak{a}}^i(M') \rightarrow f_{\mathfrak{a}}^i(M) \rightarrow f_{\mathfrak{a}}^i(M'') \rightarrow \cdots.$$

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(ii) (See [3, Theorem 1.3]). $f.\text{grade}(\mathfrak{a}, M) \leq \dim(M) - cd(\mathfrak{a}, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper $\mathcal{S}$ denotes a Serre subcategory of the category of R -modules and R -homomorphisms (we recall that a class $\mathcal{S}$ of R -modules is a Serre subcategory of the category of R -modules and R -homomorphisms if $\mathcal{S}$ is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of $\mathfrak{a}$ with respect to M in $\mathcal{S}$ as the infimum of the integers i such that $f_a^i(M) \notin \mathcal{S}$ and is denoted by $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_{\mathfrak{a}}(M)$ is a pure submodule of M , then $\text{Hom}_R(\frac{R}{\mathfrak{m}}, f_a^t(\Gamma_{\mathfrak{a}}(M)))$ and $\text{Hom}_R(\frac{R}{\mathfrak{m}}, f_a^{t-1}(\frac{M}{\Gamma_{\mathfrak{a}}(M)}))$ belong to $\mathcal{S}$, where $t = f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$.

In Section 3, we shall define the formal cohomological dimension of $\mathfrak{a}$ with respect to M in $\mathcal{S}$ as the supremum of the integers i such that $f_a^i(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\mathfrak{a}, M)$. (See definition 3.1). The main result of this section is that if $f_a^i(M) \in \mathcal{S}$ and $H_{\mathfrak{m}}^i(M) \in \mathcal{S}$ for all $i > t$, then $\frac{R}{\mathfrak{a}} \otimes_R f_a^t(M)$ belongs to $\mathcal{S}$.

2. The formal grade of a module in a Serre subcategory

Definition 2.1. The formal grade of $\mathfrak{a}$ with respect to M in $\mathcal{S}$ is the infimum of the integers i such that $f_a^i(M) \notin \mathcal{S}$ and is denoted by $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$.

Proposition 2.2. Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ be an ideal of R . If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of finitely generated R -modules, then the following statements hold.

- (a) $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M) \geq \min\{f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, L), f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, N)\}.$
- (b) $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, L) \geq \min\{f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M), f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, N) + 1\}.$
- (c) $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, N) \geq \min\{f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, L) - 1, f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)\}.$

Proof. According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \rightarrow f_a^{i-1}(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow f_a^{i+1}(L) \rightarrow \cdots.$$

So, the result follows.

Corollary 2.3. If $\underline{x} = x_1, \dots, x_n$ is a regular M -sequence, then $f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{\underline{x}M} \right) \geq f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M) - n$.

Proof. Consider the following exact sequence ($n \in \mathbb{N}$)

$$0 \rightarrow \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, \dots, x_n)M} \rightarrow 0$$

whenever $n = 1$ by $(x_1, \dots, x_{n-1})M$ we means 0.

Corollary 2.4. Let $\mathbf{a}$ and $\mathbf{b}$ be ideals of R . Then

- (a) $f.\text{grade}_{\mathcal{S}} (\mathbf{a} \cap \mathbf{b}, M) \geq \min\{f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M), f.\text{grade}_{\mathcal{S}} (\mathbf{b}, M), f.\text{grade}_{\mathcal{S}} ((\mathbf{a}, \mathbf{b}), M) + 1\}$.
- (b) $f.\text{grade}_{\mathcal{S}} ((\mathbf{a}, \mathbf{b}), M) \geq \min\{f.\text{grade}_{\mathcal{S}} (\mathbf{a} \cap \mathbf{b}, M) - 1, f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M), f.\text{grade}_{\mathcal{S}} (\mathbf{b}, M)\}$.

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \rightarrow \frac{M}{\mathbf{a}^n M \cap \mathbf{b}^n M} \rightarrow \frac{M}{\mathbf{a}^n M} \oplus \frac{M}{\mathbf{b}^n M} \rightarrow \frac{M}{(\mathbf{a}^n, \mathbf{b}^n)M} \rightarrow 0.$$

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\dots \rightarrow \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left(\frac{M}{(\mathbf{a} \cap \mathbf{b})^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left(\frac{M}{\mathbf{a}^n M} \right) \oplus \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left(\frac{M}{\mathbf{b}^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left(\frac{M}{(\mathbf{a}, \mathbf{b})^n M} \right) \rightarrow \dots$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that M is a finitely generated R -module and N_1 and N_2 are submodules of M . Then considering the exact sequence $0 \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow$

$$\frac{M}{N_1} \oplus \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0$$

- we shall have
- (a) $f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{N_1 \cap N_2} \right) \geq \min\{f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{N_1} \right), f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{N_2} \right), f.\text{grade}_{\mathcal{S}} \mathbf{a}, MN_1 + N_2 + 1\}$.
- (b) $f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{N_1 + N_2} \right) \geq \min\left\{f.\text{grade}_{\mathcal{S}} \left(\frac{M}{N_1 \cap N_2} \right) - 1, f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{N_1} \right), f.\text{grade}_{\mathcal{S}} \left(\mathbf{a}, \frac{M}{N_2} \right), f.\text{grade}_{\mathcal{S}} \mathbf{a}, MN_1, MN_2\right\}$.

Theorem 2.6. Let $\mathbf{a}$ be an ideal of a local ring $(R, \mathbf{m})$, M be a finitely generated R -module and L be a pure submodule of M . Then $f.\text{grade}_{\mathcal{S}} (\mathbf{a}, L) \geq f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M)$ where $\mathcal{S}$ is a Serre subcategory of the category of R -modules and R -homomorphisms. In particular, $\inf \{i | H_{\mathbf{m}}^i(L) \notin \mathcal{S}\} \geq \inf \{i | H_{\mathbf{m}}^i(M) \notin \mathcal{S}\}$.

Proof. Let L be a pure submodule of M . So $\frac{L}{a^n L} \rightarrow \frac{M}{a^n M}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)] , $H_m^i\left(\frac{L}{a^n L}\right) \rightarrow H_m^i\left(\frac{M}{a^n M}\right)$ is injective. Since inverse limit is a left exact functor, $f_a^i(L)$ is isomorphic to a submodule of $f_a^i(M)$. Consequently $f.\text{grade}_S(\mathbf{a}, L) \geq f.\text{grade}_S(\mathbf{a}, M)$. If $\mathbf{a} = 0$ then, $f.\text{grade}_S(0, M) = \inf \{i | H_m^i(M) \notin \mathcal{S}\}$ and the result follows.

Corollary 2.7. If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a pure exact sequence of finitely generated R -modules, then $\min \{f.\text{grade}_S(\mathbf{a}, L), f.\text{grade}_S(\mathbf{a}, N) + 1\} \geq f.\text{grade}_S(\mathbf{a}, M)$.

Proof. Since L is a pure submodules of M , as a result of the previous theorem, $f.\text{grade}_S(\mathbf{a}, L) \geq f.\text{grade}_S(\mathbf{a}, M)$. Hence we must prove that $f.\text{grade}_S(\mathbf{a}, N) + 1 \geq f.\text{grade}_S(\mathbf{a}, M)$. We assume that $i < f.\text{grade}_S(\mathbf{a}, M)$ and we show that $i < f.\text{grade}_S(\mathbf{a}, N) + 1$. Consider the following long exact sequence.

$$\cdots \rightarrow f_a^{i-1}(M) \rightarrow f_a^{i-1}(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow \cdots (**)$$

If $i < f.\text{grade}_S(\mathbf{a}, M)$, then $f_a^0(M), f_a^1(M), \dots, f_a^{i-1}(M), f_a^i(M) \in \mathcal{S}$. On the other hand, since $i < f.\text{grade}_S(\mathbf{a}, M) \leq f.\text{grade}_S(\mathbf{a}, L)$, $f_a^0(L), \dots, f_a^i(L) \in \mathcal{S}$. Hence, it follows from (**) that $f_a^0(N), \dots, f_a^{i-1}(N) \in \mathcal{S}$ and so $i - 1 < f.\text{grade}_S(\mathbf{a}, N)$.

Theorem 2.8. Let $(R, \mathbf{m})$ be a local ring, $\mathbf{a}$ be an ideal of R , $\mathcal{S}$ be a Serre subcategory of the category of R -modules and R -homomorphisms and $M \in \mathcal{S}$ be a finitely generated R -module such that $\Gamma_{\mathbf{a}}(M)$ is a pure submodule of M . Then $\text{Hom}_R\left(\frac{R}{\mathbf{a}}, f_a^t(\Gamma_{\mathbf{a}}(M))\right) \in \mathcal{S}$, where $t = f.\text{grade}_S(\mathbf{a}, M)$.

Proof. Due to the previous theorem, $f.\text{grade}_S(\mathbf{a}, \Gamma_{\mathbf{a}}(M)) \geq f.\text{grade}_S(\mathbf{a}, M)$. If $f.\text{grade}_S(\mathbf{a}, \Gamma_{\mathbf{a}}(M)) > f.\text{grade}_S(\mathbf{a}, M)$, then the result is obvious. Accordingly, we assume that $f.\text{grade}_S(\mathbf{a}, \Gamma_{\mathbf{a}}(M)) = f.\text{grade}_S(\mathbf{a}, M)$. We know that $\text{Supp}(\Gamma_{\mathbf{a}}(M)) \subseteq \text{Var}(\mathbf{a})$. By using [4, Lemma 2.3], $f_a^i(\Gamma_{\mathbf{a}}(M)) \cong H_m^i(\Gamma_{\mathbf{a}}(M))$ for all $i \geq 0$. So, if $j < f.\text{grade}_S(\mathbf{a}, M)$, then $f_a^j(\Gamma_{\mathbf{a}}(M)) \cong H_m^j(\Gamma_{\mathbf{a}}(M)) \in \mathcal{S}$ and $\text{Ext}_R^k\left(\frac{R}{\mathbf{m}}, H_m^j(\Gamma_{\mathbf{a}}(M))\right) \in \mathcal{S}$ for all $k \geq 0$ and $j < f.\text{grade}_S(\mathbf{a}, M)$. Moreover $\text{Ext}_R^t\left(\frac{R}{\mathbf{m}}, \Gamma_{\mathbf{a}}(M)\right) \in \mathcal{S}$, because $\Gamma_{\mathbf{a}}(M) \in \mathcal{S}$. Consequently, according to [7, Theorem 2.2],

$$\text{Hom}_R\left(\frac{R}{\mathbf{m}}, H_m^t(\Gamma_{\mathbf{a}}(M))\right) \in \mathcal{S}, \text{ where } t = f.\text{grade}_S(\mathbf{a}, M).$$

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in \mathcal{S}$ be a submodule of $f_a^t(\Gamma_{\mathbf{a}}(M))$, where $t = f.\text{grade}_S(\mathbf{a}, M)$. Then $\text{Hom}_R\left(\frac{R}{\mathbf{m}}, \frac{f_a^t(\Gamma_{\mathbf{a}}(M))}{X}\right) \in \mathcal{S}$.

Proof. Consider the long exact sequence:

$$Hom_R\left(\frac{R}{\underline{m}}, f_a^t(\Gamma_a(M))\right) \rightarrow Hom_R\left(\frac{R}{\underline{m}}, \frac{f_a^t(\Gamma_a(M))}{X}\right) \rightarrow Ext_R^1\left(\frac{R}{\underline{m}}, X\right). (*)$$

In accordance with the previous theorem $Hom_R\left(\frac{R}{\underline{m}}, f_a^t(\Gamma_a(M))\right) \in \mathcal{S}$. Moreover $Ext_R^1\left(\frac{R}{\underline{m}}, X\right) \in \mathcal{S}$. It follows from the exact sequence (*) that $Hom_R\left(\frac{R}{\underline{m}}, \frac{f_a^t(\Gamma_a(M))}{X}\right) \in \mathcal{S}$.

Theorem 2.10. Suppose that $\underline{a}$ is an ideal of $(R, \underline{m})$ and $M \in \mathcal{S}$ is a finitely generated R -module such that $\Gamma_a(M)$ is a pure submodule of M . Then $Hom_R\left(\frac{R}{\underline{m}}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in \mathcal{S}$, where $t = f.\text{grade}_{\mathcal{S}}(\underline{a}, M)$.

Proof. One has $f.\text{grade}_{\mathcal{S}}(\underline{a}, \Gamma_a(M)) \geq f.\text{grade}_{\mathcal{S}}(\underline{a}, M)$, by Theorem 2.6. Now, the exact sequence $0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0$ induces the following long exact sequence:

$$\dots \xrightarrow{\alpha} f_a^{t-1}(\Gamma_a(M)) \xrightarrow{\beta} f_a^{t-1}(M) \xrightarrow{\gamma} f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \xrightarrow{\xi} f_a^t(\Gamma_a(M)) \xrightarrow{\varphi} \dots. (*)$$

Using the exact sequence (*), we obtain the short exact sequence $0 \rightarrow \text{Im}(\beta) \rightarrow f_a^{t-1}(M) \rightarrow \text{Im}(\gamma) \rightarrow 0$. Since $f_a^{t-1}(M) \in \mathcal{S}$, $\text{Im}(\beta) \in \mathcal{S}$ and $\text{Im}(\gamma) \in \mathcal{S}$. Furthermore, we have the exact sequence $0 \rightarrow \text{Im}(\xi) \rightarrow H_m^t(\Gamma_a(M)) \rightarrow \text{Im}(\varphi) \rightarrow 0$ which induces the following long exact sequence:

$$0 \rightarrow Hom_R\left(\frac{R}{\underline{m}}, \text{Im}(\xi)\right) \rightarrow Hom_R\left(\frac{R}{\underline{m}}, H_m^t(\Gamma_a(M))\right) \rightarrow \dots.$$

Thus $Hom_R\left(\frac{R}{\underline{m}}, \text{Im}(\xi)\right) \in \mathcal{S}$. Finally, by considering the short exact sequence $0 \rightarrow \text{Im}(\gamma) \rightarrow f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \rightarrow \text{Im}(\xi) \rightarrow 0$ we can conclude that $Hom_R\left(\frac{R}{\underline{m}}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in \mathcal{S}$.

Theorem 2.11. Suppose that R is complete with respect to the $\underline{a}$ -adic topology and $M \in \mathcal{S}$ be a finitely generated R -module and t a positive integer such that $f_a^i(M) \in \mathcal{S}$ for all $i < t$. Then $Hom_R\left(\frac{R}{\underline{m}}, f_a^t(M)\right) \in \mathcal{S}$.

Proof. We use induction on t . Let $t=0$. Consider the following isomorphisms.

$$\begin{aligned} Hom_R\left(\frac{R}{\underline{m}}, f_a^0(M)\right) &\cong \varprojlim_{n \in \mathbb{N}} Hom_R\left(\frac{R}{\underline{m}}, H_m^0\left(\frac{M}{\underline{a}^n M}\right)\right) \cong \varprojlim_{n \in \mathbb{N}} Hom_R\left(\frac{R}{\underline{m}}, \frac{M}{\underline{a}^n M}\right) \\ &\cong Hom_R\left(\frac{R}{\underline{m}}, \varprojlim_{n \in \mathbb{N}} \left(\frac{M}{\underline{a}^n M}\right)\right) \cong Hom_R\left(\frac{R}{\underline{m}}, \hat{M}^{\underline{a}}\right) \cong Hom_R\left(\frac{R}{\underline{m}}, M\right) \end{aligned}$$

It is clear that $Hom_R(\frac{R}{\underline{m}}, M) \in \mathcal{S}$. So by the above isomorphisms, we deduce that

$$Hom_R(\frac{R}{\underline{m}}, f_a^0(M)) \in \mathcal{S}.$$

Suppose that $t > 0$ and the result is true for all integer i less than t . Set $N := \Gamma_{\underline{m}}(M)$. Then $f_a^i(M) \cong f_a^i(\frac{M}{N})$ for all $i > 0$, and so we may assume that $depth_R(M) > 0$. There is an M -regular element $x \in \underline{m}$. The exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow \frac{M}{xM} \rightarrow 0$ induces the following long exact sequence:

$$\begin{aligned} \cdots \rightarrow f_a^{t-2}(M) &\xrightarrow{x} f_a^{t-2}(\frac{M}{xM}) \xrightarrow{f} f_a^{t-2}(\frac{M}{xM}) \\ &\rightarrow f_a^{t-1}(M) \xrightarrow{x} f_a^{t-1}(\frac{M}{xM}) \xrightarrow{g} f_a^{t-1}(\frac{M}{xM}) \\ &\rightarrow f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{h} \cdots. \quad (*) \end{aligned}$$

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \rightarrow \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)} \rightarrow f_a^{t-1}(\frac{M}{xM}) \rightarrow (0 : x)_{f_a^t(M)} \rightarrow 0.$$

Now, this exact sequence induces the following long exact sequence:

$$\begin{aligned} 0 \rightarrow Hom_R\left(\frac{R}{\underline{m}}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}\right) &\rightarrow Hom_R\left(\frac{R}{\underline{m}}, f_a^{t-1}(\frac{M}{xM})\right) \rightarrow Hom_R\left(\frac{R}{\underline{m}}, (0 : x)_{f_a^t(M)}\right) \rightarrow \\ Ext_R^1\left(\frac{R}{\underline{m}}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}\right) &\rightarrow \cdots. \quad (**) \end{aligned}$$

By using $(*)$, $f_a^i(\frac{M}{xM}) \in \mathcal{S}$ for all $i < t - 1$. Therefore by the induction hypothesis $Hom_R(\frac{R}{\underline{m}}, f_a^{t-1}(\frac{M}{xM})) \in \mathcal{S}$. Furthermore $Ext_R^1(\frac{R}{\underline{m}}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}) \in \mathcal{S}$ because $f_a^{t-1}(M) \in \mathcal{S}$. Thus in accordance with $(**)$, $Hom_{f_a^t(M)}(\frac{R}{\underline{m}}, (0 : x)) \in \mathcal{S}$. Since $x \in \underline{m}$ according to [9,10.86] we have the following isomorphisms.

$$\begin{aligned} Hom_{f_a^t(M)}\left(\frac{R}{\underline{m}}, (0 : x)\right) &\cong Hom_R\left(\frac{R}{\underline{m}}, Hom_R\left(\frac{R}{xR}, f_a^t(M)\right)\right) \cong \\ Hom_R\left(\frac{R}{\underline{m}} \otimes_R \frac{R}{xR}, f_a^t(M)\right) &\cong Hom_R\left(\frac{R}{\underline{m}}, f_a^t(M)\right). \end{aligned}$$

Consequently $Hom_R(\frac{R}{\underline{m}}, f_a^t(M)) \in \mathcal{S}$.

3. The formal cohomological dimension in a Serre subcategory

We recall from [3, Theorem 1.1] that for a finitely generated R -module M , $\sup\{i \in \mathbb{N}_0 \mid f_a^i(M) \neq 0\} = \dim(\frac{M}{aM})$.

Definition 3.1. The formal cohomological dimension of M with respect to $\underline{a}$ in $\mathcal{S}$ is The supremum of the integers i such that $f_a^i(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\underline{a}, M)$.

Theorem 3.2. Suppose that $\mathcal{S}$ is a Serre subcategory of the category of R -modules and R -homomorphisms and L and N are two finitely generated R -modules such that $Supp_R(L) \subseteq Supp_R(N)$. Then $f.cd_{\mathcal{S}}(\underline{a}, L) \leq f.cd_{\mathcal{S}}(\underline{a}, N)$.

Proof. It is enough to prove that $f_a^i(L) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\underline{a}, N)$ and all finitely generated R -module L such that $Supp_R(L) \subseteq Supp_R(N)$. We use descending induction on i . For all $i > \dim(\frac{L}{aL}) + f.cd_{\mathcal{S}}(\underline{a}, N)$, $f_a^i(L) = 0 \in \mathcal{S}$. Let $i > f.cd_{\mathcal{S}}(\underline{a}, N)$ and the result is proved for $i + 1$. By Gruson's theorem, there is a chain $0 = L_0 \subset L_1 \subset \dots \subset L_l = L$ of submodules of L such that $\frac{L_i}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of N . Consider the exact sequence $0 \rightarrow L_{i-1} \rightarrow L_i \rightarrow \frac{L_i}{L_{i-1}} \rightarrow 0$ ($i = 0, 1, \dots, l$). We may assume that $l = 1$. The exact sequence $0 \rightarrow K \rightarrow \bigoplus_{j=1}^t N \rightarrow L \rightarrow 0$ where K is a finitely generated R -module induces the following long exact sequence:

$$\dots \rightarrow f_a^i(\bigoplus_{j=1}^t N) \rightarrow f_a^i(L) \rightarrow f_a^{i+1}(K) \rightarrow \dots. (*)$$

Based on the induction hypothesis $f_a^{i+1}(K) \in \mathcal{S}$. Moreover $f_a^i(\bigoplus_{j=1}^t N) = \bigoplus_{j=1}^t f_a^i(N) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\underline{a}, N)$. Hence it follows from the exact sequence (*) that $f_a^i(L) \in \mathcal{S}$.

The next example shows that even if $Supp_R(M) = Supp_R(N)$, then it may not true that $f.grade_{\mathcal{S}}(\underline{a}, M) = f.grade_{\mathcal{S}}(\underline{a}, N)$.

Example 3.3. (See [4, Example 4.3 (i)]) Let $(R, \mathfrak{m})$ be a 2 dimensional complete regular local ring, $\mathcal{S} = 0$ and $\underline{a}$ be an ideal of R with $\dim(\frac{R}{\underline{a}}) = 1$. Then by using [5, Theorem 1.1], $f.grade_{\mathcal{S}}(\underline{a}, R) = 1$ and $f.grade_{\mathcal{S}}(\underline{a}, \frac{R}{\underline{a}}) = 0$. Set $M := R \oplus \frac{R}{\underline{a}}$.

Then $Supp_R(M) = Supp_R(R)$. But

$$f.grade_{\mathcal{S}}(\underline{a}, M) = \inf\{f.grade_{\mathcal{S}}(\underline{a}, R), f.grade_{\mathcal{S}}(\underline{a}, \frac{R}{\underline{a}})\} = 0.$$

Corollary 3.4. For all $x \in \underline{a}$, $f.cd_{\mathcal{S}}(\underline{a}, M) \geq f.cd_{\mathcal{S}}(\underline{a}, \frac{M}{xM})$.

Corollary 3.5. Suppose that $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of finitely generated R -modules. Then $f.cd_{\mathcal{S}}(\underline{a}, M) = \max\{f.cd_{\mathcal{S}}(\underline{a}, L), f.cd_{\mathcal{S}}(\underline{a}, N)\}$.

Proof. Since $Supp_R(M) = Supp_R(L) \cup Supp_R(N)$ by referring to Theorem 3.2 we deduce that $f.cd_S(\mathfrak{a}, M) \geq f.cd_S(\mathfrak{a}, L)$ and $f.cd_S(\mathfrak{a}, M) \geq f.cd_S(\mathfrak{a}, N)$. Therefore $f.cd_S(\mathfrak{a}, M) \geq \max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\}$.

Next we prove that $\max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\} \geq f.cd_S(\mathfrak{a}, M)$.

Let $i > \max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\}$. Then $f_a^i(N), f_a^i(L) \in \mathcal{S}$ and from the exact sequence $f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N)$ we conclude that $f_a^i(M) \in \mathcal{S}$. Thus, $\max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\} \geq f.cd_S(\mathfrak{a}, M)$.

We recall that the cohomological dimension of an R -module M with respect to an ideal $\mathfrak{a}$ of R in $\mathcal{S}$ is defined as

$$cd_S(\mathfrak{a}, M) := \sup \{i \in \mathbb{N}_0 | H_a^i(M) \notin \mathcal{S}\}.$$

The following lemma shows that when we considering the Artinianness of $f_a^i(M)$, we can assume that M is $\mathfrak{a}$ -torsion-free.

Lemma 3.6. Suppose that $\mathfrak{a}$ is an ideal of a local ring $(R, \mathfrak{m})$ and t be a non-negative integer. If $H_{\mathfrak{m}}^i(M) \in \mathcal{S}$ for all $i \geq t$, then the following are equivalent:

- (a) $f_a^i(M) \in \mathcal{S}$ for all $i \geq t$.
- (b) $f_a^i\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Proof. According to the hypothesis $t > cd_S(\mathfrak{m}, M)$. On the other hand $Supp_R(\Gamma_{\mathfrak{a}}(M)) \subseteq Supp_R(M)$. So by referring to [7, Theorem 3.5], $cd_S(\mathfrak{m}, \Gamma_{\mathfrak{a}}(M)) \leq cd_S(\mathfrak{m}, M)$. Thus, $t > cd_S(\mathfrak{m}, \Gamma_{\mathfrak{a}}(M))$ and $H_{\mathfrak{m}}^i(\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \rightarrow f_a^i(\Gamma_{\mathfrak{a}}(M)) \rightarrow f_a^i(M) \rightarrow f_a^i\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \rightarrow f_a^{i+1}(\Gamma_{\mathfrak{a}}(M)) \rightarrow \cdots (*)$$

According to [4, Lemma 2.3] $f_a^i(\Gamma_{\mathfrak{a}}(M)) \cong H_{\mathfrak{m}}^i(\Gamma_{\mathfrak{a}}(M))$. By using the hypothesis $f_a^i(\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. So it follows from the exact sequence (*) that $f_a^i(M) \in \mathcal{S}$ if and only if $f_a^i\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Theorem 3.7. Let $(R, \mathfrak{m})$ be a local ring and $M \in \mathcal{S}$ be a finitely generated R -module of dimension d such that $cd_S(\mathfrak{m}, M) \leq f.cd_S(\mathfrak{a}, M)$. Then $\frac{f_a^t(M)}{af_a^t(M)} \in \mathcal{S}$ where $t = f.cd_S(\mathfrak{a}, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim\left(\frac{M}{af_a^t(M)}\right) = 0$. Accordingly to [3, Theorem 1.1], $f_a^i(M) = 0$ for all $i > 0$.

Moreover $f_a^0(M) \cong M \in \mathcal{S}$. By definition $H_m^i(M) \in \mathcal{S}$ for all $i > t$. Therefore from the above lemma we can assume that M is $\mathbf{a}$ -torsion-free and there is an M -regular element $x \in \mathbf{a}$. Consider the long exact sequence :

$$\cdots \rightarrow f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots (*)$$

By using the hypothesis $f_a^i(M) \in \mathcal{S}$ for all $i > t$ (because $t = f.cd_{\mathcal{S}}(\mathbf{a}, M)$). So using the above long exact sequence $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i > t$. By induction hypothesis, $\frac{f_a^t\left(\frac{M}{xM}\right)}{af_a^t\left(\frac{M}{xM}\right)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence (*) we get the following short exact sequence.

$$0 \rightarrow \text{Im}(f) \rightarrow f_a^t\left(\frac{M}{xM}\right) \rightarrow \text{Im}(g) \rightarrow 0$$

So we obtain the following long exact sequence.

$$\cdots \rightarrow \text{Tor}_I^R\left(\frac{R}{\mathbf{a}}, \text{Im}(g)\right) \rightarrow \frac{\text{Im}(f)}{a\text{Im}(f)} \rightarrow \frac{f_a^t\left(\frac{M}{xM}\right)}{af_a^t\left(\frac{M}{xM}\right)} \rightarrow \frac{\text{Im}(g)}{a\text{Im}(g)} \rightarrow 0.$$

Since $f_a^t(M) \in \mathcal{S}$ and $\text{Im}(g)$ is a submodule of $f_a^{t+1}(M)$, we deduce that $\text{Tor}_I^R\left(\frac{R}{\mathbf{a}}, \text{Im}(g)\right) \in \mathcal{S}$. On the other hand, $\frac{f_a^t\left(\frac{M}{xM}\right)}{af_a^t\left(\frac{M}{xM}\right)} \in \mathcal{S}$. Therefore, $\frac{\text{Im}(f)}{a\text{Im}(f)} \in \mathcal{S}$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{x} \frac{f_a^t(M)}{af_a^t(M)} \rightarrow \frac{\text{Im}(f)}{a\text{Im}(f)} \rightarrow 0.$$

So, $\frac{f_a^t(M)}{af_a^t(M)} \cong \frac{\text{Im}(f)}{a\text{Im}(f)}$ because $x \in \mathbf{a}$. Consequently, $\frac{f_a^t(M)}{af_a^t(M)} \in \mathcal{S}$.

Proposition 3.8. For a finitely generated R -module M ,

$$f.cd_{\mathcal{S}}(\mathbf{a}, M) = \max \{f.cd_{\mathcal{S}}\left(\mathbf{a}, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}.$$

Proof. Set $N := \bigoplus_{P \in \text{Ass}_R(M)} \frac{R}{P}$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.cd_{\mathcal{S}}(\mathbf{a}, M) = f.cd_{\mathcal{S}}(\mathbf{a}, N) = \max \{f.cd_{\mathcal{S}}\left(\mathbf{a}, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}.$

Proposition 3.9. Assume that $\mathbf{a}$ is an ideal of the local ring $(R, \mathbf{m})$. Then $\text{Hom}_R\left(\frac{R}{\mathbf{m}}, f_a^0(M)\right) \in \mathcal{S}$ if and only if. $\text{Hom}_R\left(\frac{R}{\mathbf{m}}, \widehat{M}^{\mathbf{a}}\right) \in \mathcal{S}$.

Proof. It is enough to consider the following isomorphisms

$$\begin{aligned} \text{Hom}_R\left(\frac{R}{\mathbf{m}}, f_a^0(M)\right) &\cong \lim_{n \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{\mathbf{m}}, H_m^0\left(\frac{M}{\mathbf{a}^n M}\right)\right) \cong \lim_{n \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{\mathbf{m}}, \frac{M}{\mathbf{a}^n M}\right) \\ &\cong \text{Hom}_R\left(\frac{R}{\mathbf{m}}, \lim_{n \in \mathbb{N}} \frac{M}{\mathbf{a}^n M}\right) \cong \text{Hom}_R\left(\frac{R}{\mathbf{m}}, \widehat{M}^{\mathbf{a}}\right). \end{aligned}$$

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