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Numerical solution of two-dimensional nonlinear Volterra integral equations by the Legendre polynomials

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Abstract

The main purpose of this article is to present an approximate solution for the two-dimensional nonlinear Volterra integral equations using Legendre orthogonal polynomials. First, the two-dimensional shifted Legendre orthogonal polynomials are defined and the properties of these polynomials are presented. The operational matrix of integration and the product operational matrix are introduced. These properties together with the Gauss-Legendre nodes are then utilized to transform the given integral equation to the solution of nonlinear algebraic equations. Also, an estimation of the error is presented. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

1. Introduction

The second kind of two-dimensional (2D) integral equations may arise from some problems of nonhomogeneous elasticity and electrostatics. All the mixed boundary value problems in the theory of elasticity for an inhomogeneous elastic half-space whose elastic modulus is a power function of the depth can be reduced to such an integral equation [1]. Dobner presented an equivalent formulation of the Dorboux problem as a 2D Volterra integral equation [2]. Also, 2D integral equations may arise in contact problems for bodies with complex properties [3-4]. We can also see this kind of equation in the theory of radio wave propagation, including three-dimensional local inhomogeneities [5], and in the theory of the elastic problem of axial translation of a

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rigid elliptical disc-inclusion [6], and various physical, mechanical and biological problems.

There are many works on developing and analyzing numerical methods for solving the 1D integral equations of the second kind [7-11]. But little work has been done to solve the 2D cases. The papers [12-15] are mainly concerned with numerical solution for linear 2D integral equations. Beltyukov and Kuznechikhina [16], proposed a class of explicit Rung-Kutta-type methods of order 3 for the solution of 2D nonlinear Volterra integral equations. Guaqiang et al. [17] introduced extrapolation method of iterated collocation solution for 2D nonlinear Volterra integral equation. The papers [18-19] applied the 2D differential transform for solving the 2D nonlinear Volterra integral equations. In [20], He's variational iteration method for solving nonlinear mixed Volterra-Fredholm integral equations was presented. In [21], 2D triangular functions was applied for the 2D Volterra-Fredholm integral equations. Also, Babolian et al. [22] have considered the use of the rationalized Haar functions for the numerical solution of nonlinear 2D integral equations.

In this paper, we consider the 2D nonlinear Volterra integral equations of the second kind

$$u(x,t) = f(x,t) + \int_0^t \int_0^x K(x,t,y,z) g\big(y,z,u(y,z)\big) dy\, dz, \tag{1}$$

where $u(x,t)$ is an unknown function, $f(x,t)$ is a continuous function defined on $[0,1] \times [0,1]$ and $K(x,t,y,z)$ and $g(y,z,u(y,z))$ are continuous functions, with g nonlinear in u.

We assume that the Eq. (1) has a unique solution $u(x,t)$ and will be found by an approximate solution using the properties of the 2D shifted Legendre orthogonal polynomials.

The outline of this paper is as follows: In Section 2, we discuss how to approximate two variable functions in terms of 2D shifted Legendre orthogonal functions and the operational matrix of integration and the product operational matrix are introduced. In Section 3, we give an approximate solution for (1). In Section 4, an estimation of the

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error is presented. Numerical examples are given in Section 5 to illustrate the accuracy of our method. Finally, concluding remarks are given in Section 6.

2. Properties of 2D shifted Legendre polynomials

2.1 2D shifted Legendre polynomials

The 2D shifted Legendre polynomials are defined on $[0,1] \times [0,1]$ as

$$\psi_{mn}(x,t) = p_m(2x-1)p_n(2t-1), \qquad m,n = 0,1,2,\cdots,$$

and are orthogonal with respect to weight function $\omega(x,t) = 1$ such that

$$\int_0^1 \int_0^1 \omega(x,t)\psi_{mn}(x,t)\psi_{ij}(x,t)\,dxdt = \begin{cases} \dfrac{1}{(2m+1)(2n+1)}, & i = m, j = n, \\ 0, & otherwise. \end{cases} \tag{2}$$

Here, p_m and p_n are the well-known Legendre polynomials, respectively of order m and n which are defined on the interval $[-1,1]$ and satisfy the following recursive formula [23]:

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_{m+1}(x) = \frac{2m+1}{m+1} x p_m(x) - \frac{m}{m+1} p_{m-1}(x), \quad m = 1,2,3,\cdots.$$

2.2 Function approximation

A function $u(x,t)$ defined on $[0,1] \times [0,1]$ may be expanded as

$$u(x,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}\psi_{mn}(x,t), \tag{3}$$

where

$$c_{mn} = \frac{(u,\, \psi_{mn})}{(\psi_{mn},\, \psi_{mn})},$$

in which $(.,.)$ denotes the inner product.

If the infinite series in (3) is truncated, then (3) can be written as

$$u(x,t) \cong \sum_{m=0}^{N} \sum_{n=0}^{N} c_{mn}\psi_{mn}(x,t) = C^T\psi(x,t),$$

where C and $\psi(x,t)$ are $(N+1)(N+1) \times 1$ vectors respectively given by

$$C = [c_{00}, c_{01}, \cdots, c_{0N}, c_{10}, c_{11}, \cdots, c_{1N}, \cdots\ c_{N0}, c_{N1}, \cdots, c_{NN}]^T, \tag{4}$$

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$$\psi(x,t) = [\psi_{00}(x,t), \cdots, \psi_{0N}(x,t), \psi_{10}(x,t), \cdots, \psi_{1N}(x,t), \psi_{N0}(x,t) \cdots, \psi_{NN}(x,t)]^T. \quad (5)$$

2.3 Operational matrix of integration

The integration of the vector $\psi(x,t)$ defined in (5) can be approximated by

$$\int_0^t \int_0^x \psi(x',t')dx'dt' \cong Q\psi(x,t) = (P \otimes P)\psi(x,t), \quad (6)$$

where Q is the $(N+1)^2 \times (N+1)^2$ operational matrix of integration, such that P is the$(N+1) \times (N+1)$ operational matrix of Legendre polynomials defined on [0,1] as follows [24]:

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\frac{1}{2N-1} & 0 & \frac{1}{2N-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{1}{2N+1} & 0 \end{bmatrix}.$$

In (6), $\otimes$ denotes the Kronecker product defined for two arbitrary matrices A and B as [25]

$$A \otimes B = (a_{ij}B).$$

2.4 The product operational matrix

The following property of the product of two vectors $\psi(x,t)$ and $\psi^T(x,t)$ will also be used. Let

$$\psi(x,t)\psi^T(x,t)C \cong \tilde{C}\psi(x,t), \quad (7)$$

where C is defined by (4) and $\tilde{C}$ is a $(N+1)^2 \times (N+1)^2$ product operational matrix. We have

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$$\psi(x,t)\psi^T(x,t)C = \begin{bmatrix} \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{00}(x,t) \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{01}(x,t) \\ \vdots \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{0N}(x,t) \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{10}(x,t) \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{11}(x,t) \\ \vdots \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{1N}(x,t) \\ \vdots \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{N0}(x,t) \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{N1}(x,t) \\ \vdots \\ \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\psi_{ij}(x,t)\psi_{NN}(x,t) \end{bmatrix}.$$

We put:

$$\psi_{ij}(x,t)\psi_{kl}(x,t) = \sum_{r=0}^{i+k}\sum_{s=0}^{j+l} a_{rs}\psi_{rs}(x,t), \tag{8}$$

and obtain the coefficients a_{rs} by the following manner.

Multiplying both sides of the Eq. (8) by $\psi_{mn}(x,t), m,n = 1,2,\cdots,N$ and integrating the result from 0 to1, yield

$$\int_0^1\int_0^1 \psi_{ij}(x,t)\,\psi_{kl}(x,t)\psi_{mn}(x,t)dxdt = \sum_{r=0}^{i+k}\sum_{s=0}^{j+l} a_{rs}\int_0^1\int_0^1 \psi_{rs}(x,t)\,\psi_{mn}(x,t)dxdt,$$

and using the Eq. (2) we obtain

$$\int_0^1\int_0^1 \psi_{ij}(x,t)\,\psi_{kl}(x,t)\psi_{mn}(x,t)dxdt = \frac{a_{mn}}{(2m+1)(2n+1)},$$

therefore

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$$a_{mn} = (2m+1)(2n+1)\int_0^1\int_0^1 \psi_{ij}(x,t)\,\psi_{kl}(x,t)\psi_{mn}(x,t)dxdt$$
$$= (2m+1)(2n+1)\int_0^1 p_i(2x-1)p_k(2x-1)p_m(2x-1)\,dx\int_0^1 p_j(2t-1)p_l(2t-1)p_n(2t-1)\,dt.$$

Now suppose

$$\omega_{i,k,m} = \int_0^1 p_i(2x-1)p_k(2x-1)p_m(2x-1)\,dx,\ \ i,k,m = 0,1,\cdots,N,$$

where $\omega_{i,k,m}$ can be computed easily [26], so we get

$$a_{mn} = (2m+1)(2n+1)\omega_{i,k,m}\omega_{j,l,n}.$$

Substituting a_{mn} into the Eq. (8) we have

$$\psi_{ij}(x,t)\psi_{kl}(x,t) = \sum_{m=0}^{i+k}\sum_{n=0}^{j+l}(2m+1)(2n+1)\omega_{i,k,m}\omega_{j,l,n}\psi_{mn}(x,t).$$

If we retain only the elements of $\psi(x,t)$ in the Eq. (5), then the matrix $\tilde{C}$ in the Eq. (7) is obtained as

$$\tilde{C} = \left[\tilde{C}_{ij}\right],\quad i,j = 0,1,\cdots,N, \tag{9}$$

where in the Eq. (9), $\tilde{C}_{ij}$, $i,j = 0,1,\cdots,N$ are $(N+1)\times(N+1)$ matrices given by

$$\tilde{C}_{ij} = (2j+1)\sum_{n=0}^{N} A_n\omega_{i,j,n},\qquad i,j = 0,1,\cdots,N,$$

and A_j, $j = 0,1,\cdots,N$ are $(N+1)\times(N+1)$ matrices as

$$[A_j]_{kl} = (2l+1)\sum_{m=0}^{N} c_{jm}\omega_{k,l,m},\ \ k,l = 0,1,\cdots,N.$$

To illustrate the matrix $\tilde{C}$ we choose $N = 2$ and get

$$\tilde{C} = \begin{bmatrix} \tilde{C}_{00} & \tilde{C}_{01} & \tilde{C}_{02} \\ \tilde{C}_{10} & \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{20} & \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & A_2 \\ \frac{1}{3}A_1 & A_0 + \frac{2}{5}A_2 & \frac{2}{3}A_1 \\ \frac{1}{5}A_2 & \frac{2}{5}A_1 & A_0 + \frac{2}{7}A_2 \end{bmatrix}$$

where

$$A_i = \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} \\ \frac{1}{3}c_{i1} & c_{i0} + \frac{2}{5}c_{i2} & \frac{2}{3}c_{i1} \\ \frac{1}{5}c_{i2} & \frac{2}{5}c_{i1} & c_{i0} + \frac{2}{7}c_{i2} \end{bmatrix},\quad i = 0,1,2.$$

3. Solution of the 2D nonlinear Volterra integral equation

Consider 2D Volterra integral equation as

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$$u(x,t) = f(x,t) + \int_0^t \int_0^x K(x,t,y,z) g\big(y,z,u(y,z)\big) dy\, dz. \tag{10}$$

Suppose that

$$G(x,t) = g\big(x,t,u(x,t)\big), \tag{11}$$

we approximate $G(x,t)$and $K(x,t,y,z)$ respectively as

$$G_N(x,t) = G^T \psi(x,t), \tag{12}$$

$$K_N(x,t,y,z) = K^T(x,t)\psi(y,z), \tag{13}$$

where

$$G = [g_{00}, g_{01}, \cdots, g_{0N}, g_{10}, \cdots, g_{1N}, \cdots, g_{N0}, \cdots, g_{NN}]^T,$$

$$K(x,t)$$

$$= [k_{00}(x,t), k_{01}(x,t), \cdots, k_{0N}(x,t), k_{10}(x,t), \cdots, k_{1N}(x,t), \cdots, k_{N0}(x,t), \cdots, k_{NN}(x,t)]^T,$$

so that

$$k_{ij}(x,t) = (2i+1)(2j+1) \int_0^1 \int_0^1 K(x,t,y,z)\psi_{ij}(y,z) dydz\,, \quad i,j = 0,1,\cdots,N,$$

and g_{ij},$i,j = 0,1,\cdots,N$, are unknown 2D shifted Legendre polynomials coefficients.

From the Eqs. (11)-(13) we obtain

$$u_N(x,t) = f(x,t) + \int_0^t \int_0^x K^T(x,t)\psi(y,z)\psi^T(y,z)\, G dydz. \tag{14}$$

Let

$$I(x,t) = \int_0^t \int_0^x K^T(x,t)\psi(y,z)\psi^T(y,z)\, G dydz,$$

then using the Eqs. (7) and (6) we have

$$I(x,t) = K^T(x,t)\tilde{G} Q\psi(x,t). \tag{15}$$

Substituting the Eq. (15) into the Eq. (14) we obtain

$$u_N(x,t) = f(x,t) + K^T(x,t)\widetilde{G} Q\psi(x,t). \tag{16}$$

Now from the Eqs. (11), (12) and (16) we have

$$g\big(x,t,f(x,t) + K^T(x,t)\widetilde{G} Q\psi(x,t)\big) = G^T\psi(x,t). \tag{17}$$

We collocate the Eq. (17) at $(N+1)^2$ points (x_i, t_j), $i,j = 1,2,\cdots,N+1$, as

$$g\left(x_i, t_j, f\big(x_i,t_j\big) + K^T\big(x_i,t_j\big)\widetilde{G} Q\psi\big(x_i,t_j\big)\right) = G^T\psi\big(x_i,t_j\big), \tag{18}$$

where x_i and t_j are the shifted Gauss-Legendre nodes (zeros of the$p_{N+1}(2x-1)$).

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The Eqs. (18) give $(N+1)^2$ nonlinear equations which can be solved for the elements of G using the well known Newton's iterative method. Substituting G into the

$$\tilde{u}_N(x,t) = f(x,t) + \left(\int_0^t \int_0^x K(x,t,y,z)\psi^T(y,z)dydz\right)G,$$

we find $\tilde{u}_N(x,t)$ as an approximate solution for the Eq. (10).

4. Estimation of the error

In this section, we analyze the error when a sufficeintly smooth function is expanded in terms of 2D shifted Legendre functions. Then an estimation of the error for the numerical method presented in the previous section is found assuming that $G(x,t)$ defined by (11) is a sufficiently smooth function.

We assume that $f(x,t)$ is a sufficiently smooth function on $\Omega = [0,1] \times [0,1]$, then there are real numbers M_1, M_2 and M_3, such that

$$\max_{(x,t)\in\Omega} \left|\frac{\partial^{N+1} f(x,t)}{\partial x^{N+1}}\right| \le M_1,$$
$$\max_{(x,t)\in\Omega} \left|\frac{\partial^{N+1} f(x,t)}{\partial t^{N+1}}\right| \le M_2,$$
$$\max_{(x,t)\in\Omega} \left|\frac{\partial^{2N+2} f(x,t)}{\partial x^{N+1}\partial t^{N+1}}\right| \le M_3.$$

Suppose that $p_N(x,t)$ is the interpolating polynomial to $f(x,t)$at points $(x_i,t_j), i,j = 0,1,\cdots,N$, where $x_i, i = 0,1,\cdots,N$ and $t_j, j = 0,1,\cdots,N$ are roots of degree-$(N+1)$ shifted Chebyshev polynomial in [0,1], then we have [27]

$$f(x,t) - p_N(x,t) = \frac{\partial^{N+1} f(\xi,t)}{\partial x^{N+1}(N+1)!}\prod_{i=0}^{N}(x-x_i) + \frac{\partial^{N+1} f(x,\eta)}{\partial t^{N+1}(N+1)!}\prod_{j=0}^{N}(t-t_j)$$
$$-\frac{\partial^{2N+2} f(\xi',\eta')}{\partial x^{N+1}\partial t^{N+1}[(N+1)!]^2}\prod_{i=0}^{N}(x-x_i)\prod_{j=0}^{N}(t-t_j),$$

such that, $\xi,\eta,\xi',\eta' \in [0,1]$. We get

$$|f(x,t) - p_N(x,t)| \le \frac{M_1}{(N+1)!}\left|\prod_{i=0}^{N}(x-x_i)\right| + \frac{M_2}{(N+1)!}\left|\prod_{j=0}^{N}(t-t_j)\right|$$
$$+\frac{M_3}{[(N+1)!]^2}\left|\prod_{i=0}^{N}(x-x_i)\right|\left|\prod_{j=0}^{N}(t-t_j)\right|,$$

and using the minmax theorem, we have

$$|f(x,t) - p_N(x,t)| \le \frac{M_1}{(N+1)!\,2^{2N+1}} + \frac{M_2}{(N+1)!\,2^{2N+1}} + \frac{M_3}{[(N+1)!]^2 2^{4N+2}}.$$

Therefore, we obtain

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$$|f(x,t) - p_N(x,t)| \leq \frac{\alpha}{(N+1)!\,2^{2N+1}}, \tag{19}$$

where $\alpha = M_1 + M_2 + \frac{M_3}{(N+1)!2^{2N+1}}$.

Theorem 1: Let $f_N(x,t) = {F_N}^T \psi(x,t)$, be the 2D shifted Legendre functions expansion of the real sufficiently smooth function $f(x,t)$ in Ω, where

$$F_N = [f_{00}, f_{01}, \cdots, f_{0N}, f_{10}, \cdots, f_{1N}, \cdots, f_{N0}, \cdots, f_{NN}]^T,$$

and f_{mn}, $m,n = 0,1,\cdots,N$ is defined by

$$f_{mn} = (2\mathrm{m}+1)(2\mathrm{n}+1)\int_0^1\int_0^1 f(x,t)\psi_{mn}(x,t)dxdt,$$

then there is a real number α such that

$$\|f(x,t) - f_N(x,t)\|_2 \leq \frac{\alpha}{(N+1)!\,2^{2N+1}}.$$

Moreover, if $\bar{F}_N = \left[\bar{f}_{00}, \bar{f}_{01}, \cdots, \bar{f}_{0N}, \bar{f}_{10}, \cdots, \bar{f}_{1N}, \cdots, \bar{f}_{N0}, \cdots, \bar{f}_{NN}\right]^T$ be an approximation for the 2D shifted Legendre functions coefficients vector F_N and $\bar{f}_N(x,t) = {\bar{F}_N}^T \psi(x,t)$, then there is a real number β such that

$$\left\|f(x,t) - \bar{f}_N(x,t)\right\|_2 \leq \frac{\alpha}{(N+1)!2^{2N+1}} + \beta\|F_N - \bar{F}_N\|_2. \tag{20}$$

Proof. By using definition of the best approximation [28] $f_N(x,t)$ to $f(x,t)$, we have

$$\|f(x,t) - f_N(x,t)\|_2 \leq \|f(x,t) - q_N(x,t)\|_2,$$

where, $q_N(x,t)$ is any arbitrary two-variate polynomial of degree less than or equal to N in variable x and t. Then, using (19) we get

$$\begin{aligned} {\|f(x,t) - f_N(x,t)\|_2}^2 &= \int_0^1\int_0^1 |f(x,t) - f_N(x,t)|^2 dxdt \\ &\leq \int_0^1\int_0^1 |f(x,t) - p_N(x,t)|^2 dxdt \\ &\leq \left(\frac{\alpha}{(N+1)!\,2^{2N+1}}\right)^2. \end{aligned} \tag{21}$$

Taking the 2th root from the both sides of (21) gives

$$\|f(x,t) - f_N(x,t)\|_2 \leq \frac{\alpha}{(N+1)!2^{2N+1}}. \tag{22}$$

To prove (20), we write

$$\left\|f(x,t) - \bar{f}_N(x,t)\right\|_2 \leq \|f(x,t) - f_N(x,t)\|_2 + \left\|f_N(x,t) - \bar{f}_N(x,t)\right\|_2. \tag{23}$$

Also, we have

$${\left\|f_N(x,t) - \bar{f}_N(x,t)\right\|_2}^2 = \int_0^1\int_0^1 \left|f_N(x,t) - \bar{f}_N(x,t)\right|^2 dxdt$$

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$$= \int_0^1 \int_0^1 \left| \sum_{m=0}^{N} \sum_{n=0}^{N} \left(f_{mn} - \bar{f}_{mn}\right) \psi_{mn}(x,t) \right|^2 dxdt$$

$$\leq \int_0^1 \int_0^1 \sum_{m=0}^{N} \sum_{n=0}^{N} \left|f_{mn} - \bar{f}_{mn}\right|^2 \sum_{m=0}^{N} \sum_{n=0}^{N} |\psi_{mn}(x,t)|^2 \, dxdt$$
$$= \sum_{m=0}^{N} \sum_{n=0}^{N} \left|f_{mn} - \bar{f}_{mn}\right|^2 \sum_{m=0}^{N} \sum_{n=0}^{N} \int_0^1 \int_0^1 |\psi_{mn}(x,t)|^2 dxdt$$
$$= \|F_N - \bar{F}_N\|_2{}^2 \sum_{m=0}^{N} \sum_{n=0}^{N} \|\psi_{mn}(x,t)\|_2{}^2$$
$$\leq \frac{\pi^4}{64} \|F_N - \bar{F}_N\|_2{}^2. \tag{24}$$

Taking the 2th root from the both sides of (24) gives

$$\left\|f_N(x,t) - \bar{f}_N(x,t)\right\|_2 \leq \frac{\pi^2}{8} \|F_N - \bar{F}_N\|_2. \tag{25}$$

Finally, from (22), (23) and (25), we obtain

$$\left\|f(x,t) - \bar{f}_N(x,t)\right\|_2 \leq \frac{\alpha}{(N+1)!\, 2^{2N+1}} + \beta \|F_N - \bar{F}_N\|_2,$$

where $\beta = \frac{\pi^2}{8}$.

From the Eqs.(10) and (11) we have the exact solution as

$$u(x,t) = f(x,t) + \int_0^t \int_0^x K(x,t,y,z) G(y,z) dy\, dz, \tag{26}$$

and the approximate solution is as

$$\tilde{u}_N(x,t) = f(x,t) + \int_0^t \int_0^x K(x,t,y,z) \bar{G}_N(y,z) dy\, dz, \tag{27}$$

where $\bar{G}_N(y,z) = \bar{G}_N{}^T \psi(x,t)$ and $\bar{G}_N$ is the computed vector obtained by the method presented in the previous section.

Subtracting (27) from (26), we obtain

$$|u(x,t) - \tilde{u}_N(x,t)| = \left| \int_0^t \int_0^x K(x,t,y,z) \left(G(y,z) - \bar{G}_N(y,z)\right) dydz \right|$$
$$\leq \left| \int_0^1 \int_0^1 K(x,t,y,z) \left(G(y,z) - \bar{G}_N(y,z)\right) dydz \right|$$
$$\leq \max_{(x,t,y,z)\in(\Omega\times\Omega)} |K(x,t,y,z)| \int_0^1 \int_0^1 |G(y,z) - \bar{G}_N(y,z)| dy\, dz.$$

The function $K(x,t,y,z)$ is a continuous function in $\Omega \times \Omega$, so there is a real number M such that

$$\max_{(x,t,y,z)\in(\Omega\times\Omega)} |K(x,t,y,z)| \leq M,$$

therefore, using Schwarz inequality and Theorem 1, we get

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$$|u(x,t)-\tilde{u}_N(x,t)| \le M\left(\int_0^1\int_0^1 |G(y,z)-\bar{G}_N(y,z)|^2 dy\, dz\right)^{\frac{1}{2}}$$
$$= M\|G(x,t)-\bar{G}_N(x,t)\|_2$$
$$\le \frac{\gamma}{(N+1)!\,2^{2N+1}} + \mu\|G_N-\overline{G}_N\|_2, \qquad (28)$$

where
$\gamma = M(\max_{(x,t)\in\Omega} |\frac{\partial^{N+1}G(x,t)}{\partial x^{N+1}}| + \max_{(x,t)\in\Omega} |\frac{\partial^{N+1}G(x,t)}{\partial t^{N+1}}| +$
$\frac{1}{(N+1)!2^{2N+1}} \max_{(x,t)\in\Omega} |\frac{\partial^{2N+2}G(x,t)}{\partial x^{N+1}\partial t^{N+1}}|)$,
and $\mu = M\frac{\pi^2}{8}$.

Finally, using (28) we have

$$\|u(x,t)-\tilde{u}_N(x,t)\|_2 \le \frac{\gamma}{(N+1)!\,2^{2N+1}} + \mu\|G_N-\overline{G}_N\|_2.$$

5. Illustrative examples

In this section, three numerical examples are included to demonstrate the validity and applicability of the proposed technique. In order to demonstrate the error of the method, we introduce the notation:

$$e_N(x,t) = |u(x,t)-\tilde{u}_N(x,t)|, \qquad (x,t)\in[0,1]\times[0,1],$$

where $u(x,t)$ and $\tilde{u}_N(x,t)$ are the exact and approximate solutions respectively.

Example 5.1. Consider a nonlinear 2D Volterra integral equation of the form [19]

$$u(x,t) = x^2e^t + \frac{1}{14}x^7 - \frac{1}{14}x^7e^{2t} - \frac{1}{5}x^5t + \int_0^t\int_0^x (y^2+e^{-2z})u^2(y,z)dydz, \quad (29)$$

with the exact solution $u(x,t) = x^2e^t$.

We applied the method presented in this paper and solved the Eq. (29). Numerical results are presented in Table 1 and Figure 1. Table 1 shows the error $e_N(x,t)$ at some points together with the results obtained by the method of [19]. It shows that by increasing$N$, the accuracy of the solution increases and the presented method is more accurate than the method of [19] for the 2D Volterra integral equations by using 2D differential transform.

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Table 1. The absolute error $e_N(x,t)$ at some points for Example 5.1.

(x,t)	Present Method with $N=4$	Present Method with $N=6$	Present Method with $N=8$	Method of [19] with $N=10$
(0.1,0.7)	2.58615×10^{-10}	2.4912×10^{-13}	4.1823×10^{-16}	5.2589×10^{-12}
(0.2,0.3)	1.7163×10^{-10}	1.3565×10^{-11}	2.5110×10^{-14}	1.82059×10^{-15}
(0.3,0.9)	2.5287×10^{-9}	3.2264×10^{-11}	7.8045×10^{-14}	7.6453×10^{-10}
(0.4,1)	1.5986×10^{-10}	5.9868×10^{-16}	2.4096×10^{-17}	4.3700×10^{-9}
(0.5,0.8)	2.0342×10^{-7}	3.0919×10^{-11}	2.4013×10^{-12}	5.7619×10^{-10}
(0.6,1)	5.0704×10^{-10}	2.9434×10^{-14}	2.0193×10^{-16}	9.8325×10^{-9}
(0.7,0.6)	1.0426×10^{-6}	1.0777×10^{-9}	6.9357×10^{-12}	4.6869×10^{-11}
(0.8,1)	4.1502×10^{-9}	3.7477×10^{-13}	1.2102×10^{-15}	1.7480×10^{-8}
(0.9,0.5)	1.0227×10^{-5}	4.4669×10^{-8}	1.2249×10^{-10}	1.0337×10^{-11}
(1,1)	3.9974×10^{-8}	3.4360×10^{-12}	5.8841×10^{-15}	2.7312×10^{-8}

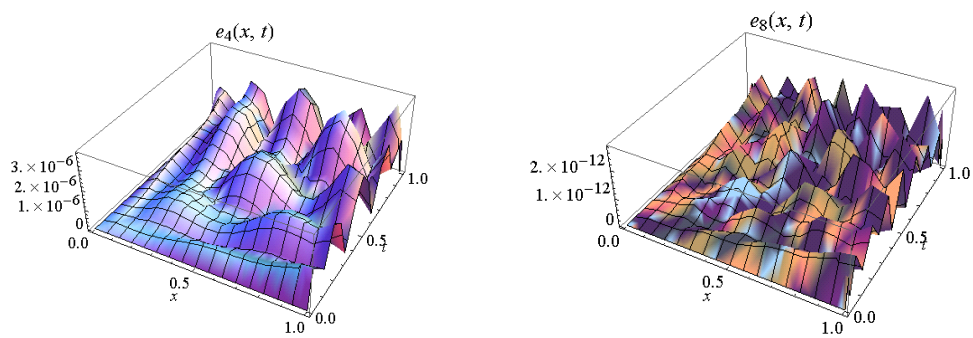

Figure 1. Graph of the $e_N(x,t)$ with $N=4,8$ for Example 5.1.

Example 5.2. Consider the nonlinear 2D Volterra integral equation [22]

$$u(x,t)=f(x,t)+\int_0^t\int_0^x (xy^2+\cos(z))u^2(y,z)dy\,dz, \tag{30}$$

where

$$f(x,t)=x\sin(t)\left(1-\frac{1}{9}x^2sin^2(t)\right)+\frac{1}{10}x^6\left(\frac{1}{2}\sin(2t)-t\right),$$

and has the exact solution $u(x,t)=x\sin(t)$.

The proposed method was applied to approximate the solution of the Eq. (30). Table 2 shows the error $e_N(x,t)$ at some points together with the results obtained by the method of [22] using 2D Haar functions.

Example 5.3. Consider the following 2D nonlinear Volterra integral equation

$$u(x,t)=(x+t)(e^x+e^t-e^{x+t})+\int_0^t\int_0^x (x+t)e^{u(y,z)}dy\,dz,$$

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where the exact solution is $u(x,t) = x + t$. Table 3 and Figure 2 illustrate the numerical results for this example.

Table 2: Numerical results for Example 5.2.

$(x,t) = (\frac{1}{2^l}, \frac{1}{2^l})$	Present Method with $N = 4$	Present Method with $N = 8$	Method of [22] with $m = 32$
$l = 1$	5.5×10^{-7}	6.7×10^{-12}	1.4×10^{-2}
$l = 2$	5.8×10^{-8}	1.8×10^{-13}	7.9×10^{-3}
$l = 3$	9.1×10^{-10}	1.2×10^{-14}	4.1×10^{-3}
$l = 4$	6.1×10^{-10}	1.1×10^{-15}	2.2×10^{-3}
$l = 5$	9.5×10^{-11}	3.9×10^{-16}	1.2×10^{-3}
$l = 6$	1.5×10^{-11}	9.7×10^{-17}	9.3×10^{-9}

Table 3: Numerical results for Example 5.3.

$(x,t) = (\frac{1}{2^l}, \frac{1}{2^l})$	$N = 2$	$N = 4$	$N = 8$	$N = 16$
$l = 1$	1.2×10^{-3}	2.31×10^{-6}	1.6×10^{-12}	4.4×10^{-16}
$l = 2$	3.8×10^{-5}	4.2×10^{-7}	7.8×10^{-14}	5.5×10^{-17}
$l = 3$	4.2×10^{-5}	1.1×10^{-8}	1.0×10^{-14}	2.7×10^{-17}
$l = 4$	8.7×10^{-6}	1.5×10^{-8}	1.1×10^{-14}	2.7×10^{-17}
$l = 5$	1.3×10^{-6}	3.7×10^{-9}	1.2×10^{-15}	2.1×10^{-17}
$l = 6$	1.8×10^{-7}	6.1×10^{-10}	6.3×10^{-16}	3.4×10^{-18}

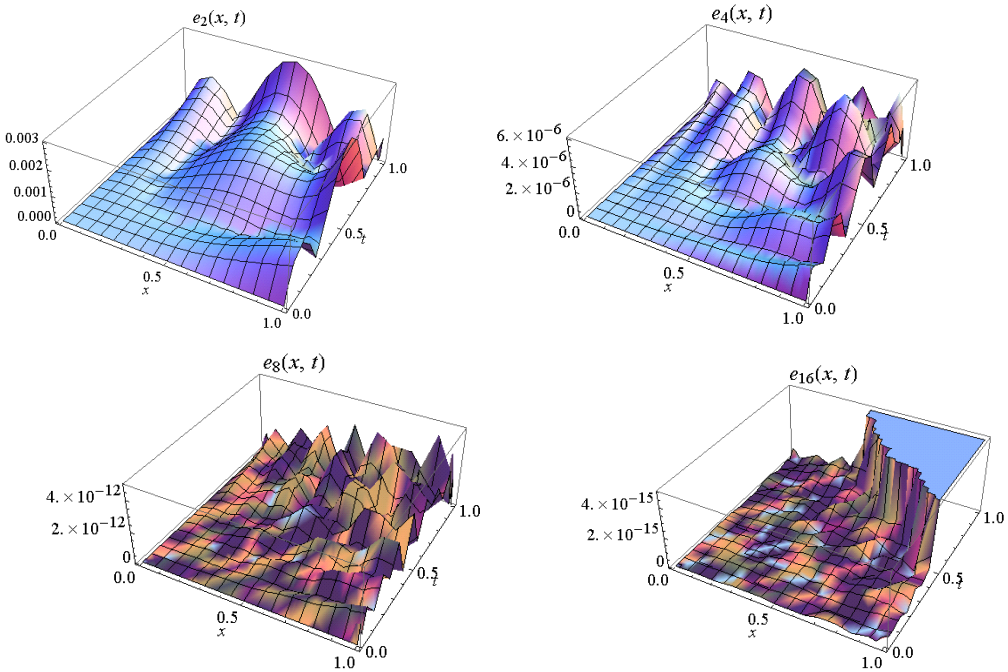

Figure 2. Graph of the $e_N(x,t)$ with $N = 2, 4, 8, 16$ for Example 5.3.

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6. Conclusion

In this paper, we presented a highly accurate method to solve the 2D nonlinear Volterra integral equations. The properties of the 2D shifted Legendre orthogonal polynomials together with the Gauss-Legendre nodes were used to transform the given problem to the solution of non-linear algebraic equations. It should be Noted that the final nonlinear equations were solved using the Newton's iterative method. We applied the presented method on three test problems and compared the results with their exact solution in order to demonstrate the validity and applicability of the method. The results obtained by the technique in the current paper were more accurate than the results reported with other methods.

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