Approximate Solution of Nonlinear Fredholm Integro-Differential Equations with time delay by using Taylor Method

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Abstract

This paper presents an appropriate numerical method to solve nonlinear Fredholm integro-differential equations with time delay. Its approach is based on the Taylor expansion. This method converts the integro-differential equation and the given conditions into the matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Taylor expansion coefficients, so that the solution of this system yields the Taylor expansion coefficients of the solution function. Then, the performance of the method is evaluated with some examples.

1. Introduction

There are limited methods for solving nonlinear Fredholm integro-differential equations with time delay. Many authors have presented numerical methods for solving integro-differential equations, for example, Galerkin method [3], Rationalized Haar functions method [4], Varitional Iteration method [5], Walsh series method [6], Semi orthogonal spline wavelets method [7] and Taylor collocation method [8]. Taylor collocation method is a numerical technique that depends on Taylor expansion. This method for Fredholm integral equations was presented by Kenwall and Liu [9], and was used for solving differential equations, difference equations, differential-difference equations, Pantograph equations [10] and integral equations [11-13]. Then, this method was extended by Sezer to Fredholm integro-differential equations [14] and linear Fredholm integro-differential equations with time delay.

Keywords: Nonlinear Fredholm integro-differential equations, Time delay, Taylor

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In this paper, the Taylor polynomials are used for approximating the solution of nonlinear Fredholm integro-differential equations with time delay

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x)+\sum_{r=0}^{l} P_r^*(x)y^{(r)}(x-\tau)=f(x)+\int_a^b K(x,t)[y(t-\tau)]^n\,dt; \quad \tau\geq 0,\ a\leq x\leq b, \quad (1)$$

that has a unique solution, with the mixed conditions:

$$\sum_{k=0}^{m-1}[a_{ik}\ y^{(k)}(a)+b_{ik}\ y^{(k)}(b)+c_{ik}\ y^{(k)}(c)]=\lambda_i, \qquad i=0,1,\dots,m-1,\ \ a\leq c\leq b, \qquad (2)$$

where a_{ik}, b_{ik} , c_{ik}, and λ_i are given real constants and $P_k(x)$, $P_r^*(x)$, $K(x,t)$ and $f(x)$ are given functions. In this paper we express the mentioned known functions by truncated Taylor expansion of degree N at $x=c$ and also consider unknown function $y(x)$ about c point of degree N as

$$y(x)\simeq\sum_{n=0}^{N}\frac{y^{(n)}(c)}{n!}(x-c)^n, \quad a\leq c\leq b,\ N\geq m. \qquad (3)$$

2. Matrix relations

Considering the Eq. (1) and finding the truncated Taylor expansion of degree N of each function in the equation at $x=c$ and their matrix representations, we first assume the solution $y(x)$ and express in the matrix form:

$$y(x)\simeq X^T M_0 Y, \qquad (4)$$

where

$$X=[1,\ (x-c),\ (x-c)^2,\dots,(x-c)^N]^T,$$

$$M_0=\begin{bmatrix} \frac{1}{0!} & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{1!} & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{2!} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{N!}\end{bmatrix}, \qquad Y=\begin{bmatrix} y^{(0)}(c)\\ y^{(1)}(c)\\ y^{(2)}(c)\\ \vdots\\ y^{(N)}(c)\end{bmatrix}.$$

Now we assume function $P_0(x)\,y^{(0)}(x)$ of differential part of Eq. (1) that is truncated Taylor expansion of degree N at $x=c$ in the form:

$$P_0(x)\,y^{(0)}(x)\simeq\sum_{n=0}^{N}\frac{1}{n!}[P_0(x)\,y^{(0)}(x)]_{x=c}^{(n)}(x-c)^n, \qquad (5)$$

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By applying the Leibnitz's rule:

$$[P_0(x)\,y^{(0)}(x)]^{(n)}_{x=c} \simeq \sum_{i=0}^{n} \binom{n}{i} P_0^{(n-i)}(c)\,y^{(i)}(c),$$

relation (5) becomes:

$$P_0(x)\,y^{(0)}(x) \simeq \sum_{n=0}^{N}\sum_{i=0}^{n} \frac{1}{n!}\binom{n}{i} P_0^{(n-i)}(c)\,y^{(i)}(c)(x-c)^n.$$

Therefore, it has the matrix form:

$$P_0(x)y^{(0)}(x) \simeq X^T P_0 Y, \tag{6}$$

where

$$P_0 = \begin{bmatrix} \frac{P_0^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{P_0^{(1)}(c)}{1!0!} & \frac{P_0^{(0)}(c)}{0!1!} & 0 & \cdots & 0 \\ \frac{P_0^{(2)}(c)}{2!0!} & \frac{P_0^{(1)}(c)}{1!1!} & \frac{P_0^{(0)}(c)}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{P_0^{(N)}(c)}{N!0!} & \frac{P_0^{(N-1)}(c)}{(N-1)!1!} & \frac{P_0^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{P_0^{(0)}(c)}{0!N!} \end{bmatrix}.$$

In a similar way, we will have:

$$P_k(x)\,y^{(k)}(x) \simeq X^T P_k Y, \quad k = 1,2,\ldots,m, \tag{7}$$

where

$$P_k = \begin{bmatrix} 0 & \ldots & 0 & \frac{P_k^{(0)}(c)}{0!0!} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \frac{P_k^{(1)}(c)}{1!0!} & \frac{P_k^{(0)}(c)}{0!1!} & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \frac{P_k^{(2)}(c)}{2!0!} & \frac{P_k^{(1)}(c)}{1!1!} & \frac{P_k^{(0)}(c)}{0!2!} & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \frac{P_k^{(N-k)}(c)}{(N-k)!0!} & \frac{P_k^{(N-k-1)}(c)}{(N-k-1)!1!} & \frac{P_k^{(N-k-2)}(c)}{(N-k-2)!2!} & \ldots & \frac{P_k^{(1)}(c)}{1!(N-k-1)!} & \frac{P_k^{(0)}(c)}{0!(N-k)!} \\ 0 & \ldots & 0 & \frac{P_k^{(N-k+1)}(c)}{(N-k+1)!0!} & \frac{P_k^{(N-k)}(c)}{(N-K)!1!} & \frac{P_k^{(N-k-1)}(c)}{(N-k-1)!2!} & \ldots & \frac{P_k^{(2)}(c)}{2!(N-k-1)!} & \frac{P_k^{(1)}(c)}{0!(N-k)!} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \frac{P_k^{(N-1)}(c)}{(N-1)!0!} & \frac{P_k^{(N-2)}(c)}{(N-2)!1!} & \frac{P_k^{(N-3)}(c)}{(N-3)!2!} & \ldots & \frac{P_k^{(k)}(c)}{k!(N-k-1)!} & \frac{P_k^{(k-1)}(c)}{(k-1)!(N-k)!} \\ 0 & \ldots & 0 & \frac{P_k^{(N)}(c)}{N!0!} & \frac{P_k^{(N-1)}(c)}{(N-1)!1!} & \frac{P_k^{(N-2)}(c)}{(N-2)!2!} & \ldots & \frac{P_k^{(k+1)}(c)}{(k+1)!(N-k-1)!} & \frac{P_k^{(k)}(c)}{k!(N-k)!} \end{bmatrix}. \tag{8}$$

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Now in a similar way, we consider Taylor expansion of degree N function $P_r^*(x)\, y^{(r)}(x-\tau)$ of differential part of Eq. (1) at $x=c$, that is in the form:

$$P_r^*(x)\, y^{(r)}(x-\tau) \simeq \sum_{n=0}^{N} \frac{1}{n!}[P_r^*(x)\, y^{(r)}(x-\tau)]_{x=c}^{(n)}(x-c)^n, \qquad r=0,1,\ldots,l. \quad (9)$$

By applying the Leibnitz's rule, its substitution in expression (9) becomes:

$$P_r^*(x)\, y^{(r)}(x-\tau) \simeq \sum_{n=0}^{N}\sum_{i=0}^{n} \frac{1}{n!}\binom{n}{i} P_r^{*(n-i)}(c)\, y^{(i)}(c-\tau)(x-c)^n,$$

Therefore, it has the matrix form:

$$P_r^*(x)\, y^{(r)}(x-\tau) \simeq X^T P_r^* Y_\tau, \quad r=0,1,\ldots,l, \quad (10)$$

where

$$Y_\tau = \left[\, y^{(0)}(c-\tau), y^{(1)}(c-\tau),\ldots, y^{(N)}(c-\tau)\right]^T. \quad (11)$$

And P_r^* can be obtained by substituting the quantities $P_r^{*(j)}(c)$ instead of $P_k^{(j)}(c)$ in the matrix P_k.

Now by substituting $(x-\tau)$ for x in Eq. (3) and using derivative, we will have:

$$\begin{aligned}
y^{(0)}(x-\tau) &\simeq \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-\tau-c)^n,\\
y^{(1)}(x-\tau) &\simeq \sum_{n=1}^{N} \frac{y^{(n)}(c)}{(n-1)!}(x-\tau-c)^{n-1},\\
y^{(2)}(x-\tau) &\simeq \sum_{n=2}^{N} \frac{y^{(n)}(c)}{(n-2)!}(x-\tau-c)^{n-2},\\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12)\\
y^{(N)}(x-\tau) &\simeq \sum_{n=N}^{N} \frac{y^{(n)}(c)}{(n-N)!}(x-\tau-c)^{n-N}.
\end{aligned}$$

System (12) writes in the matrix form:

$$Y_\tau \simeq X_\tau Y\ , \quad (13)$$

where X_τ is as follows:

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$$X_\tau = \begin{bmatrix} \frac{1}{0!} & \frac{(-\tau)^1}{1!} & \frac{(-\tau)^2}{2!} & \cdots & \frac{(-\tau)^N}{N!} \\ 0 & \frac{1}{0!} & \frac{(-\tau)^1}{1!} & \cdots & \frac{(-\tau)^{N-1}}{(N-1)!} \\ 0 & 0 & \frac{1}{0!} & \cdots & \frac{(-\tau)^{N-2}}{(N-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!} \end{bmatrix}.$$

Now by substituting Eq. (13) for Eq. (10), we get the following matrix representation:

$$P_r^*(x)\, y^{(r)}(x-\tau) \simeq X^T P_r^* Y_\tau = X^T P_r^* X_\tau Y, \tag{14}$$

Also $f(x)$ and $k(x,t)$ are approximated by the Taylor expansion of degree N as:

$$f(x) \simeq X^T M_0 F, \qquad K(x,t) \simeq X^T K T, \tag{15}$$

where

$$F = [f^{(0)}(c), f^{(1)}(c), \ldots, f^{(N)}(c)]^T, \quad K = (k_{nm})^T_{(N+1)\times(N+1)}, \quad k_{nm} = \frac{1}{m!n!}\frac{\partial^{m+n}K(c,c)}{\partial x^m \partial t^n},$$

$$m,n = 0,1,\ldots,N.$$

and

$$T = [1,(t-c),(t-c)^2,\ldots,(t-c)^N]^T.$$

Now by using the Cauchy product of n series, the matrix form of function $y(x)$, is written as:

$$y^n(x) \simeq \overline{X}_n^T \overline{Y}_n, \tag{16}$$

where

$$\overline{Y}_n = \begin{bmatrix} \overline{y}_0^n \\ \overline{y}_1^n \\ \vdots \\ \overline{y}_{nN}^n \end{bmatrix} = \begin{bmatrix} \sum_{k_1=0}^{0}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{n-1}=0}^{k_{n-2}} \frac{y_{k_{n-1}} y_{k_{n-2}-k_{n-1}} \cdots y_{k_1-k_2} y_{k_1}}{(k_{n-1})!(k_{n-2}-k_{n-1})!\ldots(k_1-k_2)!(0-k_1)!} \\ \sum_{k_1=0}^{1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{n-1}=0}^{k_{n-2}} \frac{y_{k_{n-1}} y_{k_{n-2}-k_{n-1}} \cdots y_{k_1-k_2} y_{k_1-k_1}}{(k_{n-1})!(k_{n-2}-k_{n-1})!\ldots(k_1-k_2)!(1-k_1)!} \\ \vdots \\ \sum_{k_1=0}^{nN}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{n-1}=0}^{k_{n-2}} \frac{y_{k_{n-1}} y_{k_{n-2}-k_{n-1}} \cdots y_{k_1-k_2} y_{k_{nN}-k_1}}{(k_{n-1})!(k_{n-2}-k_{n-1})!\ldots(k_1-k_2)!(nN-k_1)!} \end{bmatrix},$$

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that $y_m = 0$ for $m > N$, and

$$\overline{X}_n = [1, (x-c), (x-c)^2, ..., (x-c)^{nN}]^T,$$

Therefore,

$$y^n(t-\tau) \simeq \overline{T}_n^T(t-\tau)\overline{Y}_n, \tag{17}$$

where

$$\overline{T}_n(t-\tau) = [1,\ (t-\tau-c),\ (t-\tau-c)^2, ..., (t-\tau-c)^{nN}]^T.$$

Now, for integral part, we apply relations (15) and (17):

$$\int_a^b K(x,t)\, y^n(t-\tau)\, dt \simeq \int_a^b X^T\, K\, T\, \overline{T}_n^T(t-\tau)\overline{Y}_n\, dt = X^T\, K\, H\overline{Y}_n, \tag{18}$$

as if:

$$H = \int_a^b T\overline{T}_n^{\,T}(t-\tau)dt = [h_{ij}], \quad h_{ij} = \frac{(b-\tau-c)^{i+j+1} - (a-\tau-c)^{i+j+1}}{i+j+1}, \qquad i = 0,1,\ldots,N,\ \ j = 0,1,\ldots,nN,$$

By putting matrix relations (6), (7), (14), (15) and (18) into Eq. (1), we get:

$$(\sum_{k=0}^{m} P_k + \sum_{r=0}^{l} P_r^* X_\tau)Y - (KH)\overline{Y}_n = M_0 F. \tag{19}$$

Now we get the matrix representation of the mixed conditions (2). By using relation (12), we find the matrix form $y^{(k)}$ at the point a, b and c as:

$$y^{(k)}(a) \simeq P^T M_k Y, \tag{20}$$

$$y^{(k)}(b) \simeq Q^T M_k Y, \tag{21}$$

$$y^{(k)}(c) \simeq R^T M_k Y, \tag{22}$$

as if:

$$P = [1,\ (a-c),\ (a-c)^2, ..., (a-c)^N]^T,$$

$$Q = [1,\ (b-c),\ (b-c)^2, ..., (b-c)^N]^T,$$

$$R = [1,\ 0,\ 0, \ldots, 0]^T,$$

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$$M_k = \begin{bmatrix} 0 & 0 & \dots & \frac{1}{0!} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{1!} & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{1}{(N-m)!} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)} .$$

By putting the matrix relations (20), (21) and (22) into Eq. (2), we obtain:

$$\sum_{k=0}^{m-1}\{a_{ik}P^T + b_{ik}Q^T + c_{ik}R^T\}M_k Y = [\lambda_i]. \quad i = 0,1,\dots,m-1,$$

Let us define U_i as:

$$U_i = \sum_{k=0}^{m-1}\{a_{ik}P^T + b_{ik}Q^T + c_{ik}R^T\}M_k = [u_{i0}, u_{i1}, \dots, u_{iN}]. \quad i = 0,1,\dots,m-1, \tag{23}$$

Thus, the matrix form of conditions (2) becomes:

$$U_i Y = [\,\lambda_i\,]. \tag{24}$$

3. Method of solution

Let us consider the matrix representation Eq. (1) in the form of relation (19). We, then, define:

$$WY - V\overline{Y}_n = M_0 F, \tag{25}$$

where

$$W = [w_{ij}] = (\sum_{k=0}^{m} P_k + \sum_{r=0}^{l} P_r^* X_\tau), \qquad i, j = 0,1,\dots,N.$$

$$V = [v_{pq}] = K\,H, \quad p, q = 0,1,\dots,N.$$

Eq. (25) can be written in the form of:

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$$[W;V:M_0F]=\begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & v_{0,0} & v_{0,1} & \cdots & v_{0,N} & ; & \frac{f^{(0)}(c)}{0!} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & ; & v_{1,0} & v_{1,1} & \cdots & v_{1,N} & ; & \frac{f^{(1)}(c)}{1!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N,0} & w_{N,1} & \cdots & w_{N,N} & ; & v_{N,0} & v_{N,1} & \cdots & v_{N,N} & ; & \frac{f^{(N)}(c)}{N!} \end{bmatrix}. \tag{26}$$

We now consider the matrix Eq. (24) that corresponds to conditions (2), in the following form:

$$[U_i;\lambda_i]=[u_{i0},u_{i1},\ldots,u_{iN};\lambda_i], \qquad i=0,1,\ldots,m-1. \tag{27}$$

To obtain the solution of Eq. (1) under conditions (2), we replace the rows of matrix (27) by the last m rows of the matrix (26). So we will have the following matrix:

$$\left[\tilde{W};\tilde{V}:\tilde{F}\right]=\begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & v_{0,0} & v_{0,1} & \cdots & v_{0,N} & ; & \frac{f^{(0)}(c)}{0!} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & ; & v_{1,0} & v_{1,1} & \cdots & v_{1,N} & ; & \frac{f^{(1)}(c)}{1!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & v_{N-m,0} & v_{N-m,1} & \cdots & v_{N-m,N} & ; & \frac{f^{(N-m)}(c)}{(N-m)!} \\ u_{0,0} & u_{0,1} & \cdots & u_{0,N} & ; & 0 & 0 & \cdots & 0 & ; & \lambda_0 \\ u_{1,0} & u_{1,1} & \cdots & u_{1,N} & ; & 0 & 0 & \cdots & 0 & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & 0 & 0 & \cdots & 0 & ; & \lambda_{m-1} \end{bmatrix}. \tag{28}$$

After solving nonlinear system:

$$\widetilde{W}Y-\widetilde{V}\overline{Y}_n=\widetilde{F}, \tag{29}$$

where

$$\widetilde{F}=[\frac{f^{(0)}(c)}{0!},\frac{f^{(1)}(c)}{1!},\ldots,\frac{f^{(N-m)}(c)}{(N-m)!},\lambda_0,\lambda_1,\ldots,\lambda_{m-1}]^T.$$

using mathematical software version 5.1, and applying Newton-Raphson iterative method, the unknown Taylor coefficients $y^{(n)}(x)$ for $n=0,1,...,N$ are obtained. When it is placed into the relation (3), the approximation solution $y(x)$ is found.

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4. Evaluating the method with numerical examples

Example 1. Let us consider the nonlinear Fredholm integro-differential equation:

$$-xy''(x)+xy'(x)+y(x)+y'(x-1)+y(x-1)=12x-\frac{31}{6}+\int_0^1 t[y(t-1)]^2\,dt,$$

with conditions $y(0)=-1, y'(0)=4$. Then, the exact solution is $y(x)=4x-1$.

In this example, we have:

$$P_0(x)=1, P_1(x)=x, P_2(x)=-x,\ P_0^*(x)=1, P_1^*(x)=1, f(x)=12x-\tfrac{31}{6}, K(x,t)=t, \tau=1.$$

Nonlinear system of relation (25) is:

$$[P_0+P_1+P_2+(P_0^*+P_1^*)X_\tau]Y-[KH]\overline{Y}_2=M_0F.$$

By applying the method of part (3), with $N=1$, we get:

$$y_0=-1, y_1=4.$$

By placing these coefficients into Eq. (3), the solution $y(x)=4x-1$, which is the same exact solution, is obtained.

Example 2. Consider the third-order nonlinear Fredholm integro-differential equation with the following mixed conditions:

$$y^{(3)}(x)+y(x)=f(x)+\int_0^2[y(t-1)]^2\,dt,$$

$$y(0)=0,\ y'(0)=0,\ y''(0)=2.$$

It has the exact solution of $y(x)=x^4-2x^3+x^2$, and:

$$f(x)=x^4-2x^3+x^2+24x-\tfrac{4516}{315}.$$

By applying the method of part (3), with $N=4$, we will have:

$$y_0=0,\ y_1=0,\ y_2=2,\ y_3=-12,\ y_4=24.$$

So the approximate solution is $y(x)=x^4-2x^3+x^2$, which is the same exact solution.

Example 3. Consider the second-order nonlinear Fredholm integro-differential equation:

$$-y''(x)+y'(x)=-1+xCosx+\frac{Sin2}{2}+Sinx+\int_0^2[y(t-1)]^2\,dt,$$

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with mixed conditions $y(0)=0, y'(0)=1$,which has the exact solution of $y(x)=\sin(x)$. We will have the same result, selecting $N=4$ and applying the method of part (3):

$$y(x) \simeq x-\frac{x^3}{3!}+\frac{x^5}{5!}-0.000137x^7.$$

Example 4. Consider the nonlinear Fredholm integro-differential equation:

$$-xy''(x)+xy'(x)+y(x)+y'(x-1)=f(x)+\int_0^1 x^3[y(t-1)]^2dt,$$

with mixed conditions $y(0)=1, y'(0)=1$ and $f(x)=e^{x-1}+e^x-\frac{x^3(e^2-1)}{2e^2}$.

By applying the method of part (3), with $N=6$, the following approximate solution is:

$$y(x) \simeq 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+0.0412x^4+0.00801x^5+0.000718x^6.$$

We are aware that the exact solution of the problem is:

$$y(x)=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots.$$

Example 5. Consider the forth-order nonlinear Fredholm integro-differential equation:

$$x^2y^{(3)}(x)+\sin(x)y(x)+y^{(4)}(x-1)=f(x)+\int_{-1}^1 e^{-2x}[y(t-1)]^2dt,$$

with mixed conditions $y(0)=-1, y'(0)=0, y''(0)=3, y'''(0)=0$, where:

$$f(x)=-\sin(x)\cos(x)-\cos(x-1)-e^{-2x}(\frac{37}{5}-8\cos(2)-4\sin(2)+\frac{\sin(4)}{4}).$$

By applying the method of part (3), with $N=8$, the approximate solution will be in the following form:

$$y(x) \simeq -1+\frac{3}{2}x^2-0.0415931x^4+0.0000472x^5+0.0011305x^6+0.0000021x^7-0.0000267x^8.$$

The exact solution of this problem is in the form:

$$y(x)=x^2-\cos(x)=-1+\frac{3}{2}x^2-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\ldots.$$

Let us suppose $e(x)=|y(x)-\tilde{y}(x)|$, where approximate $y(x)$ is $\tilde{y}(x)$. So the related error can be seen in table (1). It should be mentioned that $e(0)=0$.

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Table (1). Error of example 5.

x	-1	-0.8	-0.6	-0.4	-0.2
$e(x)$	2.36×10^{-4}	5.39×10^{-5}	6.28×10^{-6}	3.36×10^{-7}	8.60×10^{-8}
x	0.2	0.4	0.6	0.8	1
$e(x)$	1.16×10^{-7}	1.31×10^{-6}	1.17×10^{-6}	2.20×10^{-5}	1.37×10^{-4}

5. Conclusion

It is usually difficult to solve nonlinear Fredholm integro-differential equations. The proposed method to obtain a solution for nonlinear Fredholm integro-differenial equations with time delay proves to be a simple and useful method. An important property of this method is that we get exact solutions in many cases. The accuracy of the obtained solution of the method can be improved by taking more terms in the Taylor expansion.

This method is applicable in different types of integro and integro-differential equations. Also, it might be applied for solving the following nonlinear Fredholm integro-differential equations with time delay:

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) + \sum_{r=0}^{l} P_r^*(x) y^{(r)}(x-\tau) = f(x) + \int_a^b K(x,t)[y(t)]^p \, dt + \int_a^b K^*(x,t)[y(t-\tau)]^q \, dt.$$

where p and q are positive integers.

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