A Collocation Method for Solving Nonlinear Differential Equations via Hybrid of Rationalized Haar Functions

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Abstract

Hybrid of rationalized Haar functions are developed to approximate the solution of the differential equations. The properties of hybrid functions which are the combinations of block-pulse functions and rationalized Haar functions are first presented. These properties together with the Newton-Cotes nodes are then utilized to reduce the differential equations to the solution of algebraic equations. The method is computationally attractive, and applications are demonstrated through illustrative examples.

1. Introduction

Many different bases functions have been used to estimate the solution of differential equations, such as orthogonal bases, Walsh functions [1], block-pulse functions [2] rationalized Haar functions [3], Laguerre functions [4], Shifted Legendre polynomial [5], Shifted Chebyshev polynomial [6] and Fourier series.

The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. Orthogonal functions have also been proposed to solve linear differential equations. These techniques have been presented, among other by Chen and Hsiao [1], Hwang and Shih [2], Razzaghi and Razzaghi [7], Razzaghi and Ordokhani [8], Ordokhani [9]. In using orthogonal rationalized Haar functions to get good accuracy the number of variable be very large as $k = 2^{\alpha+1}$, $\alpha = 0,1,2,\ldots$. So for decrease variable and time and higher accuracy we use idea of hybrid of block-pulse and rationalized Haar (HRH) functions to approximate differential equations.

Keywords: Hybrid; Rationalized Haar functions; Block-pulse functions; Newton-Cotes; Differential equations; Nonlinear; Operational matrix; Orthogonal functions

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In this paper we present a new direct computational method for solving ordinary second differential equations. The method consists of reducing the problem to set of algebraic equations by first $\ddot{y}$ as a HRH functions with unknown coefficient.

The operational matrix of integration and Newton Cotes nodes [10] $t_p = \frac{2p-1}{2Nk}$, $p = 1, 2, 3, \ldots Nk$ are then utilized to evaluate the coefficients of the HRH functions. Since the integration of the hybrid of block-pulse and rationalized Haar functions vector are continuous functions, we obtained continuous approximate solution for $y(t)$.

The paper is organized as follows: Section 2 is devoted to the basic formulation of the hybrid functions of block-pulse and rationalized Haar functions required for our subsequent development, the operational matrices of integration are also derived. In Section 3 we apply the proposed numerical method to the numerical solution of ordinary differential equations, and in Section 4 we report our numerical finding and demonstrate the accuracy of the proposed method.

2. Properties of Hybrid Functions

2.1. Hybrid functions

The HRH functions $\phi_{nr}(t), n = 1, 2, \ldots, N,\ r = 0, 1, \ldots, k-1,\ k = 2^{\alpha+1},\ \alpha = 0, 1, 2, \ldots$ defined on [0,1), have three arguments, r and n are the order for rationalized Haar functions and block-pulse functions respectively and t is the normalized time and is defined as

$$\phi_{nr}(t) = \begin{cases} \phi_r(Nt + 1 - n), & \frac{n-1}{N} \le t \le \frac{n}{N} \\ 0, & otherwise. \end{cases} \tag{1}$$

Here, $\phi_r(t) = RH(r,t)$ are the well-known rationalized Haar functions of order r which are orthogonal in the interval $[0,1)$ and satisfy the following formula [11]:

$$RH(r,t) = \begin{cases} 1, & J_1 \le t < J_{1/2} \\ -1, & J_{1/2} \le t < J_0 \\ 0, & otherwise \end{cases}$$

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where

$$J_u = \frac{j-u}{2^i}, \quad u = 0, \frac{1}{2}, 1.$$

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0,1,2,3,\ldots \quad j = 1,2,3,\ldots,2^i.$$

$RH(0,t)$ is defined for $i = j = 0$ and is given by

$$RH(0,t) = 1, \quad 0 \le t < 1.$$

Since $\phi_{nr}(t)$ is the combination of rationalized Haar functions and block-pulse functions which are both complete and orthogonal, thus the set of hybrid functions are complete orthogonal set.

The orthogonality property given by

$$\int_0^1 \phi_{nr}(t)\phi_{n'r'}(t)dt = \begin{cases} \dfrac{2^{-i}}{N}, & n = n', r = r' \\ 0, & otherwise \end{cases}$$

where

$$r = 2^i + j + 1, \quad r' = 2^{i'} + j' + 1.$$

2.2. Function approximation

A function $f(t)$ defined over $[0,1)$ may be expanded in HRH functions as

$$f(t) = \sum_{r=0}^{\infty}\sum_{n=1}^{\infty} a_{nr}\phi_{nr}(t), \tag{2}$$

where a_{nr} are given by

$$a_{nr} = \frac{(f, \phi_{nr})}{\|\phi_{nr}\|^2} = 2^i N \int_0^1 f(t)\phi_{nr}(t)dt$$

and $(.,.)$ denotes the inner product. If the infinite series in Eq. (2) is truncated, then Eq. (2) can be written as.

$$\mathrm{f(t)} = \sum_{\mathrm{r=0}}^{\mathrm{k-1}}\sum_{\mathrm{n=1}}^{\mathrm{N}} \mathrm{a_{nr}}\Phi_{\mathrm{nr}}(\mathrm{t}) = \mathrm{A^T B(t)}.$$

The HRH functions coefficient vector A and HRH functions vector $B(t)$ are defined as

$$A = [a_{10}, a_{11}, \ldots, a_{1k-1} \mid a_{20}, a_{21}, \ldots, a_{2k-1} \mid \ldots \mid a_{N0}, a_{N1}, \ldots, a_{Nk-1}]^T, \tag{3}$$

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and

$$B(t)=[\phi_{10}(t),\phi_{11}(t),\ldots,\phi_{1k-1}(t)\mid\phi_{20}(t),\phi_{21}(t),\ldots,\phi_{2k-1}(t)\mid\ldots\mid\phi_{N0}(t),\phi_{N1}(t),\ldots,\phi_{Nk-1}(t)]^T. \quad (4)$$

Also, the integration of the cross,product of two hybrid vector is

$$\int_0^1 B(t)B^T(t)dt=W=\frac{1}{N}diag.(D,D,\ldots,D), \quad (5)$$

where W is the $Nk\times Nk$ matrix and

$$D=\frac{1}{N}diag.(1,1,\frac{1}{2},\frac{1}{2},\underbrace{\frac{1}{2^2},\ldots,\frac{1}{2^2}}_{2^2},\ldots,\underbrace{\frac{1}{2^\alpha},\frac{1}{2^\alpha},\ldots,\frac{1}{2^\alpha}}_{2^\alpha}).$$

2.3. Operational matrix of integration

The integration of the vector $B(t)$ defined in Eq. (4) is given by

$$\int_0^t B(t')dt'=PB(t), \quad (6)$$

where P is the $Nk\times Nk$ operational matrix for integration and is given by

$$P=\frac{1}{N}\begin{bmatrix}\hat{P} & H & H & \cdots & H & H\\ O & \hat{P} & H & \cdots & H & H\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ O & O & O & \cdots & \hat{P} & H\\ O & O & O & \cdots & O & \hat{P}\end{bmatrix}.$$

Also H is the $k\times k$ matrix represented by

$$H=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ . & . & . & \cdots & .\\ . & . & . & \cdots & .\\ . & . & . & \cdots & .\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$$

and $\hat{P}$ is the $k\times k$ operational matrix for rationalized Haar functions and is given in [9]

as
$$\hat{P}=\hat{P}_{k\times k}=\frac{1}{2k}\begin{pmatrix}2k\hat{P}_{\frac{k}{2}\times\frac{k}{2}} & -\hat{\Phi}_{\frac{k}{2}\times\frac{k}{2}}\\ -\hat{\Phi}^{-1}_{\frac{k}{2}\times\frac{k}{2}} & O\end{pmatrix},$$

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where $\hat{P}_{1\times1}=[\frac{1}{2}], \hat{\Phi}_{1\times1}=[1]$ and $\hat{\Phi}_{k\times k}$ is given by

$$\hat{\Phi}_{k\times k}=[\Phi(\frac{1}{2k}),\Phi(\frac{3}{2k}),\ldots,\Phi(\frac{2k-1}{2k})],$$

with

$$\Phi(t)=[\phi_0(t),\phi_1(t),\ldots,\phi_{k-1}(t)]^T.$$

3. Solution of Ordinary Differential Equation

Consider the following second order ordinary differential equations

$$f(t,y,\dot{y},\ddot{y})=0, \quad t\in[0,1) \tag{7}$$

$$y(0)=y_0, \quad \dot{y}(0)=y_1, \tag{8}$$

where f is known analytic function. To solve Eq. (7) with initial conditions in Eq. (8) we let

$$\ddot{y}(t)=\sum_{r=0}^{k-1}\sum_{n=1}^{N}a_{nr}\Phi_{nr}(t)=A^TB(t). \tag{9}$$

Using Eqs. (6) and (9) we get

$$\dot{y}(t)=A^T\int_0^t B(t')dt'+y_1=A^TPB(t)+y_1, \tag{10}$$

similarity for $y(t)$ we have

$$\begin{aligned} y(t)&=A^TP\int_0^t B(t')dt'+y_1t+y_0\\ &=A^TP^2B(t)+y_1+y_0. \end{aligned} \tag{11}$$

Since $\int_0^t B(t')dt'$ is continuous vector we get a continuous solution for $y(t)$.

By substituting Eqs. (9)-(11) in Eq. (7), we obtain

$$f(t,A^TP^2B(t)+y_1t+y_0,A^TPB(t)+y_1,A^TB(t))=0. \tag{12}$$

we now collocate Eq. (12) at Nk points t_p as

$$f(t_p,A^TP^2B(t_p)+y_1t_p+y_0,A^TPB(t_p)+y_1,A^TB(t_p))=0. \tag{13}$$

For a suitable collocation points we choose Newton-Cotes nodes as [10]

$$t_p=\frac{2p-1}{2Nk}, \quad p=1,2,3,\ldots,Nk.$$

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Equation (13) gives Nk nonlinear equations which can be solved for the elements of A in Eq. (3) using Newton's iterative method. Ultimately the continuous approximate solution $y(t)$ is following as

$$y(t) = A^T P \int_0^t B(t')dt' + y_1 t + y_0 = A^T P^2 B(t) + y_1 t + y_0.$$

Illustrative Examples

Example 1. Consider the error function [11]

$$\ddot{y}(t) + 2t\dot{y}(t) = 0, \quad t \in [0,1); \quad y(0) = 0, \quad \dot{y}(0) = \frac{2}{\sqrt{\pi}}, \tag{14}$$

where

$$f(t, y(t), \dot{y}(t), \ddot{y}(t)) = \ddot{y}(t) + 2t\dot{y}(t), \quad y_0 = 0, \quad y_1 = \frac{2}{\sqrt{\pi}}.$$

By using the method in section 3 Eq. (14) is solved. The computational result for k=4,N=2 together with the exact solution $y(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-x^2)dx$ and method of [11] are given in Table 1.

Table 1. Approximate and exact solutions for example 1

t	Exact	Method of [11] for $k = 8$	Present method for $k = 4,\ N = 2$
0	0	0	0
0.2	0.2227026	0.22268	0.2227024
0.4	0.4283924	0.42837	0.4283921
0.6	0.6038561	0.60384	0.6038560
0.8	0.7421010	0.74208	0.7421013
1	0.8427008	0.84269	0.8427004

Example 2. Airy's differential equation

Consider linear differential equation

$$\ddot{y}(t) + ty(t) = 0.$$

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Solutions of Airy's equation are called Airy's functions, and have applications to the theory diffraction [12]. We solve Airy's equation for initial condition $y(0)=0,\ \dot{y}(0)=1,$ as

$$\ddot{y}(t)+ty(t)=0,\quad t\in[0,1);\quad y(0)=0,\quad \dot{y}(0)=1. \tag{15}$$

A solution known of the Eq. (15) is given by [12]

$$y(t)=t+\sum_{q=1}^{\infty}\frac{(-1)^q t^{3q+1}}{4\times 7\times 10\times\cdots(3q+1)3^q(q!)}.$$

Here, we solve the Eq. (15) by using the present method. The computational result for $k=8, N=1$ and $k=4, N=2$ together with the exact solution are given in Table 2.

Table 2. Approximate and exact solutions for example 2

t	Exact	Present method for $k=8, N=1$	Present method for $k=4, N=2$
0	0	0	0
0.2	0.1998667	0.19983	0.1998663
0.4	0.3978699	0.39784	0.3978695
0.6	0.5892554	0.58922	0.5892550
0.8	0.7662804	0.76625	0.7662800
1	0.9186289	0.91865	0.9186294

Example 3. Consider nonlinear ordinary differential equation [11]

$$\ddot{y}(t)+\dot{y}(t)+y^3(t)=2+4t^2+t^6,\quad t\in[0,1];\quad y(0)=0,\quad \dot{y}(0)=0. \tag{16}$$

In this example we have

$$f(t,y,\dot{y},\ddot{y})=\ddot{y}(t)+\dot{y}(t)+y^3(t)-2-4t^2-t^6,\quad y_0=y_1=0.$$

By using Eq. (13) we can solve Eq. (16) for the unknown vector A. The computational result for $k=4, N=2,$ together with the exact solution $y(t)=t^2$ and method [11] are given in Table 3.

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Table 3. Approximate and exact solutions for example 3

t	Exact	Method of [11]	Present method for $k=4, N=2$
0	0	0	0
0.2	0.04	0.04003	0.0400002
0.4	0.16	0.16003	0.1600005
0.6	0.36	0.36002	0.3600004
0.8	0.64	0.64001	0.6400007
1	1	1.0000	0.1000002

Example 4. Lane-Emden's differential equation

We consider the nonlinear initial value problem given in [13] by

$$t\ddot{y}(t)+6\dot{y}(t)+14ty(t)=-4ty(t)\ln y(t), \quad t\in[0,1]; \quad y(0)=1, \ \dot{y}(0)=0. \qquad (17)$$

We applied the method presented in this paper and solved Eq. (17) with $k=8, N=1$, and $k=4, N=2$. The computational result together with the exact solution $y(t)=\exp(-t^2)$ are given in Table 4.

2. CONCLUSION

In the present work the HRH functions are used to solve the solution of nonlinear ordinary differential equations. The HRH functions together Newton-Cotes nodes t_p solution of Eq. (10) is converted to a problem of solving a system of algebraic equations. The matrix P have many zeros; hence this method is much faster than rationalized Haar functions. In this method time and computations are small. Illustrative examples are given to demonstrate the validity and applicability of the proposed method.

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Table 4. Approximate and exact solutions for example 4

t	Exact	Present method for $k=8, N=1$	Present method for $k=4, N=2$
0	1	1	1
0.2	0.960789	0.9605	0.960786
0.4	0.852144	0.8520	0.852142
0.6	0.697676	0.6973	0.697673
0.8	0.527292	0.5274	0.527295
1	0.367879	0.3680	0.367883

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References

1. C. F. Chen and C. H. Hasio, A Walsh series direct methods for solving variational problems, Journal of the Franklin Institute, 300 (1975) 265-280.
2. C. H. Wang and Y. P. Shih, Explicit solutions of integral equations via block-pulse functions, Int. J. Syst. Sci., 13 (1982) 773-782.
3. M. Ohkita and Y. Koyabashi, An application of rationalized Haar functions to solution of linear differential equations, IEEE Trans, Circuit, 9 (1986) 853-862.
4. C. Hwang and Y. P. Shih, Solution of integral equations via Laguerre polynomials, Comp. and Elect. Engng., 9 (1982) 123-129.
5. R. Y. Chang and M. L. Wang, Shifted Legendre direct method for variational problems, Journal of Optimization Theory and Applications, 39 (1983) 299-307.
6. I. R. Horng and J. H. Chon, Shifted Chebyshev direct method for solving variational Problems, International Journal of Systems Science, 16 (1985) 855-861.
7. M. Razzaghi and M. Razzaghi, Fourier series direct method for variational problems, International Journal of Control, 48 (1988) 887-895.

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8. M. Razzaghi and Y. Ordokhani, Solution of nonlinear Volterra-Hammerstein integral equations via rationalized Haar functions, Mathematical Problems in Engineering, 7 (2001) 205-218.
9. Y. Ordokhani, Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions, Applied Mathematics and Computation, 180 (2006) 436-443.
10. G. M. Phillips and P. J. Taylor, Theory and application of numerical analysis, Academic Press , New York, 1973.
11. M. Razzaghi and Y.Ordokhani, Solution of differential equations via rationalized Haar Functions, Mathematics and Computers in Simulation, 56 (2000) 235-246.
12. G. F. Simmons, Differential equations with applications and historical notes, Tata Mcgraw-Hill, New Dehli, 1983.
13. A. M. Wazwaz, A new method for solving singular initial value problem in the second order ordinary differential equations, Applied Mathematics and computation, 128 (2002) 45-57.

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